EXPLICIT ESTIMATES FOR POLYNOMIAL SYSTEMS DEFINING IRREDUCIBLE SMOOTH COMPLETE INTERSECTIONS

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Abstract. This paper deals with properties of the algebraic variety defined as the set of zeros of a "typical" sequence of polynomials. We consider various types of "nice" varieties: set-theoretic and ideal-theoretic complete intersections, absolutely irreducible ones, and nonsingular ones. For these types, we present a nonzero "genericity" polynomial of explicitly bounded degree in the coefficients of the sequence that vanishes if its variety is not of the type. Here, the number of polynomials and their degrees are fixed. Over finite fields, this yields bounds on the number of such sequences. We also show that most sequences (of at least two polynomials) define a degenerate variety, namely an absolutely irreducible nonsingular hypersurface in some linear projective subspace.

1. Introduction

Over a field K, a sequence $\boldsymbol{f} = (f_1, \ldots, f_s)$ of homogeneous polynomials in $n+1$ variables with $n > s$ defines a projective variety $V \subseteq \mathbb{P}^n_K$, namely, its set of common roots. Intuitively, most such sequences are regular and V enjoys "nice" properties, such as being a set-theoretic or ideal-theoretic complete intersection, being (absolutely) irreducible, and nonsingular. This paper confirms this intuition in a quantitative way.

For a fixed pattern $(d_1, \ldots, d_s)$ of degrees $d_i = \deg f_i$, the set of all such $\boldsymbol{f}$ forms a multiprojective space in a natural fashion. For properties as above, we provide a nonzero "genericity polynomial" P of explicitly bounded degree in variables corresponding to the coefficients in $\boldsymbol{f}$ such that any $\boldsymbol{f}$ with $P(\boldsymbol{f}) \neq 0$ enjoys the property. Thus "most" sequences define a nice variety.

If K is finite with q elements, we obtain as a consequence bounds on the probability that the variety is nice. They have the form $1 - O(q^{-1})$ with explicit constants depending on the geometric data, but not on q.

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A case of interest is to fix the geometric data and consider growing q. Then these bounds are increasing with q

For each property, we first present a genericity polynomial as above that works for any field. From this, we derive numerical estimates in the case of finite fields.

Section 2 provides some notational background. In Sections 3 and 4 we fix the degree sequence of our polynomial sequence and study four geometric properties of the corresponding projective variety in the appropriate projective space: being a set-theoretic or an ideal-theoretic complete intersection, absolute irreducibility and nonsingularity. For these properties, we present a nonzero "genericity" polynomial of bounded degree in variables corresponding to the coefficients of the polynomial sequence that vanishes if the corresponding variety is not of the type; see the "geometric" Theorems 3.2, 3.5, 4.5 and 4.2. These results show that a typical sequence of polynomials is regular and defines an ideal-theoretic complete intersection which is absolutely irreducible and nonsingular.

We then apply the bounds to polynomial sequences over finite fields to obtain numerical results, which may also be interpreted as probabilities for sequences chosen uniformly at random. Let $\mathbb{F}_q$ be the finite field with q elements, where q is a prime power. Multivariate polynomial systems over $\mathbb{F}_q$ arise in connection with many fundamental problems in cryptography, coding theory, or combinatorics; see, e.g., Wolf & Preneel (2005), Ding *et al.* (2006), Cafure *et al.* (2012), Cesaratto *et al.* (2014), Matera *et al.* (2014). A random multivariate polynomial system over $\mathbb{F}_q$ with more equations than variables is likely to be unsolvable over $\mathbb{F}_q$. On the other hand, when there are more variables than equations, the system is likely to be solvable over $\mathbb{F}_q$ (see Fusco & Bach (2009) for the phase transition between these two regimes).

Further information can be obtained if the projective variety $V \subset \mathbb{P}^n_{\mathbb{F}}$ defined by $f_1, \ldots, f_s$ possesses "nice" geometric properties. The projective variety V is the set of common zeros of $f_1, \ldots, f_s$ in the n–dimensional projective space $\mathbb{P}^n_{\mathbb{F}}$ over an algebraic closure $\mathbb{F}$ of $\mathbb{F}_q$. Indeed, if V is known to be a nonsingular or an absolutely irreducible complete intersection, then estimates on the deviation from the expected number of points of V in $\mathbb{P}^n(\mathbb{F}_q)$ are obtained in Deligne (1974), Hooley (1991), Ghorpade & Lachaud (2002), Cafure *et al.* (2015), Matera *et al.* (2016). This motivates the study of the "frequency" with which such geometric properties arise.

Over finite fields, the geometric theorems plus an appropriate version of the Weil bound yield bounds on the number of such sequences of polynomials; see Corollaries 3.7, 4.6 and 4.3. This can be interpreted as probabilities for polynomial sequences chosen uniformly at random. The lower bounds tend to 1 with growing field size.

For $s = 1$, the variety defined by a single polynomial $f_1 \in K[X_0, \ldots, X_n]$ is a hypersurface, which is absolutely irreducible if the polynomial f_1 is. Counting irreducible multivariate polynomials over a finite field is a classical subject which goes back to the works of Carlitz (1963), Carlitz (1965) and Cohen (1968/1969); see Mullen & Panario (2013), Section 3.6, for further references. In von zur Gathen *et al.* (2013), exact formulas on the number of absolutely irreducible multivariate polynomials over a finite field and easy–to–use approximations are provided. No results on the number of sequences of polynomials $f_1, \ldots, f_s$ over a finite field defining an absolutely irreducible projective variety are known to the authors.

Concerning nonsingularity over an arbitrary field K, the set of all s–tuples of homogeneous polynomials $f_1, \ldots, f_s \in K[X_0, \ldots, X_n]$ of degrees $d_1, \ldots, d_s$ defining a projective variety which fails to be nonsingular of dimension $n - s$ is called the *discriminant locus*. It is well-known that the discriminant locus is a hypersurface of the space of s–tuples $f_1, \ldots, f_s$ of homogeneous polynomials of degrees $d_1, \ldots, d_s$; see, e.g., Gel'fand *et al.* (1994) for the case of the field of complex numbers. This hypersurface is defined by a polynomial in the coefficients of the polynomials $f_1, \ldots, f_s$ which is homogeneous in the coefficients of each f_i. For $s = 1$, a well-known result of George Boole asserts that the discriminant locus has degree $(n + 1)(d_1 - 1)^n$; see Cayley (1845). On the other hand, in Benoist (2012) an exact formula for the degrees of the discriminant locus is provided. The calculation is based on a study of dual varieties of nonsingular toric varieties in characteristic zero. Then the case of positive characteristic is dealt with using projective duality theory. Our approach is based on the analysis of an incidence variety with tools of classical projective geometry. We do not obtain exact formulas, but easy–to–use approximations for the homogeneity degrees.

The above results assume a fixed sequence of degrees. When we vary the degrees, it is natural to keep the Bézout number $\delta = d_1 \cdots d_s$ constant. In Section 5, we show that "most" polynomial sequences define a degenerate variety, namely, a hypersurface in some linear projective subspace. Here, "most" refers to the dimension of the set of all relevant polynomial sequences for infinite K, and to their number in the case of finite K.

Let $d_1, \ldots, d_s \geq 1$ be given and let $f_1, \ldots, f_s \in K[X_0, \ldots, X_n]$ be homogeneous polynomials of degrees $d_1, \ldots, d_s$ with coefficients in an arbitrary field K. A basic quantity associated to $f_1, \ldots, f_s$ is the Bézout number $\delta = d_1 \cdots d_s$. For example, for $K = \mathbb{F}_q$ the cost of several algorithms for finding a common zero of $f_1, \ldots, f_s$ with coefficients in $\mathbb{F}_q$ is measured in terms of the Bézout number δ (see, e.g., Huang & Wong (1999), Cafure & Matera (2006), Bardet *et al.* (2013)). In this sense, it may be interesting to study geometric properties that can be expected from a typical sequence $f_1, \ldots, f_s$ of $K[X_0, \ldots, X_n]$ for which only the Bézout number δ is given. For a given degree pattern with Bézout number δ, the results of the first part of this paper show that the corresponding projective variety is expected to be a complete intersection of dimension $n-s$ and degree δ. Therefore, the situation is somewhat reminiscent of that of the Chow variety of projective varieties of a given dimension and degree in a given projective space.

The Chow variety of curves of $\mathbb{P}^n_{\overline{K}}$ of degree δ over an algebraic closure $\overline{K}$ of a field K is considered in Eisenbud & Harris (1992). It is shown that its largest irreducible component consists of planar irreducible curves provided that δ is large enough. Over a finite field, Cesaratto *et al.* (2013) use this to obtain estimates, close to 1, on the probability that a uniformly random curve defined over a finite field $\mathbb{F}_q$ is absolutely irreducible and planar. The present paper shows that for a fixed Bézout number, a typical sequence of polynomials with corresponding degree pattern defines an irreducible hypersurface V in some linear projective subspace of $\mathbb{P}^n_{\overline{K}}$ (Theorem 5.7). Thus the points of V span a linear space of dimension $1 + \dim V$, which is the minimal value unless V is linear. In particular, a typical V is degenerate. Here, "typical" refers to the dimension of the set of polynomial sequences, fixing the Bézout number. Furthermore, for a finite field we provide nearly optimal bounds on the number of polynomial sequences that define such degenerate varieties. This result generalizes the corresponding one of Cesaratto *et al.* (2013) from curves to projective varieties of arbitrary dimension.

2. Notions and notations

We collect some basic definitions and facts, using standard notions and notations of algebraic geometry, which can be found in, e.g., Kunz (1985) or Shafarevich (1994). The reader familiar with this material may want to skip ahead to Section 3.

Let K be a field, $\overline{K}$ an algebraic closure, and $\mathbb{P}^n_{\overline{K}}$ the n–dimensional projective space over $\overline{K}$. It is endowed with its Zariski topology over $\overline{K}$, for which a closed set is the zero locus of homogeneous polynomials of $\overline{K}[X_0, \ldots, X_n]$. We shall also consider the Zariski topology of $\mathbb{P}^n_{\overline{K}}$ over K, where closed sets are zero loci of homogeneous polynomials in $K[X_0, \ldots, X_n]$.

A subset $V \subset \mathbb{P}^n_{\overline{K}}$ is a *projective K-variety* if it is the set $Z(f_1, \ldots, f_s)$ (or $\{f_1 = 0, \ldots, f_s = 0\}$) of common zeros in $\mathbb{P}^n_{\overline{K}}$ of a family $f_1, \ldots, f_s \in K[X_0, \ldots, X_n]$ of homogeneous polynomials.

A K-variety $V \subset \mathbb{P}^n_{\overline{K}}$ is *K-irreducible* if it cannot be expressed as a finite union of proper K-subvarieties of V. Further, V is *absolutely irreducible* if it is $\overline{K}$–irreducible as a $\overline{K}$–variety. Any K-variety V can be expressed as a non-redundant union $V = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_r$ of irreducible (absolutely irreducible) K-varieties, unique up to reordering, which are called the *irreducible* (*absolutely irreducible*) *K-components* of V.

For a K-variety $V \subset \mathbb{P}^n_{\overline{K}}$, its *defining ideal* $I(V)$ is the set of polynomials of $K[X_0, \ldots, X_n]$ vanishing on V. The *coordinate ring* $K[V]$ of V is defined as the quotient ring $K[X_0, \ldots, X_n]/I(V)$. The *dimension* $\dim V$ of a K-variety V is the length m of a longest chain $V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_m$ of nonempty irreducible K-varieties contained in V. A K-variety V is called *equidimensional* if all irreducible K-components of V are of the same dimension m; then V is of *pure dimension m*.

The *degree* $\deg V$ of an irreducible K-variety V is the maximum number of points lying in the intersection of V with a linear space L of codimension $\dim V$, for which $V \cap L$ is finite. More generally, following Heintz (1983) (see also Fulton (1984)), if $V = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_r$ is the decomposition of V into irreducible K-components, then the degree of V is

$$\deg V = \sum_{1 \le i \le r} \deg \mathcal{C}_i.$$

The following *Bézout inequality* holds (see Heintz (1983), Fulton (1984), Vogel (1984)): if V and W are K-varieties, then

$$\deg(V \cap W) \le \deg V \cdot \deg W. \tag{2.1}$$

Let $V \subset \mathbb{P}^n_{\overline{K}}$ be a projective variety and $I(V) \subset \overline{K}[X_0, \ldots, X_n]$ its defining ideal. For $x \in V$, the *dimension* $\dim_x V$ *of V at x* is the maximum of the dimensions of the irreducible components of V that contain x. If $I(V) = (f_1, \ldots, f_s)$, a point $x \in V$ is called *regular* if the rank of the Jacobian matrix $(\partial f_i/\partial X_j)_{1 \le i \le s, 0 \le j \le n}(x)$ of $f_1, \ldots, f_s$ with respect to $X_0, \ldots, X_n$ at x is equal to $n - \dim_x V$. Otherwise, the point x is called *singular*. The set of singular points of V is the *singular locus* $\mathrm{Sing}(V)$ of V. A variety is called *nonsingular* if its singular locus is empty.

2.1. **Complete intersections.** If the projective K-variety $V = Z(f_1, \ldots, f_s)$ defined by homogeneous polynomials $f_1, \ldots, f_s$ in $K[X_0, \ldots, X_n]$ is of pure dimension $n - s$, it is a *set-theoretic complete intersection* (defined over K). This is equivalent to the sequence $(f_1, \ldots, f_s)$ being a *regular sequence*, meaning that f_1 is nonzero and each f_i is neither zero nor a zero divisor in $K[X_0, \ldots, X_n]/(f_1, \ldots, f_{i-1})$ for $2 \le i \le s$. In particular, any permutation of a regular sequence of homogeneous polynomials is also regular.

If the ideal $(f_1, \ldots, f_s)$ generated by $f_1, \ldots, f_s$ is radical, then we say that V is an *ideal-theoretic complete intersection*, or simply a *complete intersection* (defined over K). The "radical" property rules out repeated components and is the appropriate notion from an algebraic point of view. If $V \subset \mathbb{P}^n_{\overline{K}}$ is a complete intersection defined over K, of dimension $n-s$ and degree δ, and $f_1, \ldots, f_s$ is a system of homogeneous generators of $I(V)$, the degrees $d_1, \ldots, d_s$ depend, up to permutation, only on V and not on the system of generators (see, e.g., Ghorpade & Lachaud (2002), Section 3). Arranging the d_i in such a way that $d_1 \geq d_2 \geq \cdots \geq d_s$, we call $\boldsymbol{d} = (d_1, \ldots, d_s)$ the *multidegree* of V.

According to the Bézout inequality (2.1), if $V \subset \mathbb{P}^n_{\overline{K}}$ is a complete intersection defined over K of multidegree $\boldsymbol{d} = (d_1, \ldots, d_s)$, then $\deg V \leq d_1 \cdots d_s$. Actually, a much stronger result holds, namely, the *Bézout theorem*:

$$\deg V = d_1 \cdots d_s. \tag{2.2}$$

See, e.g., Harris (1992), Theorem 18.3, or Smith *et al.* (2000), §5.5, page 80.

In what follows we shall deal with a particular class of complete intersections, which we now define. A K-variety is *regular in codimension* m if the singular locus $\mathrm{Sing}(V)$ of V has codimension at least $m+1$ in V, namely if $\dim V - \dim \mathrm{Sing}(V) \geq m+1$. A complete intersection V which is regular in codimension 1 is called *normal*; actually, normality is a general notion that agrees on complete intersections with the one we use here. A fundamental result for projective complete intersections is the Hartshorne connectedness theorem (see, e.g., Kunz (1985), Theorem VI.4.2), which we now state. If $V \subset \mathbb{P}^n_{\overline{K}}$ is a set-theoretic complete intersection defined over K and $W \subset V$ is any K-subvariety of codimension at least 2, then $V \setminus W$ is connected in the Zariski topology of $\mathbb{P}^n_{\overline{K}}$ over K. For a normal set-theoretic complete intersection V defined over $\overline{K}$, the subvariety $W = \mathrm{Sing}(V) \subset V$ has codimension at least 2. Then the Hartshorne connectedness theorem asserts that $V \setminus W$ is connected, which implies that V is absolutely irreducible.

The next statement summarizes several well-known relations among the concepts introduced above.

Fact 2.1. *For a projective variety $V \subset \mathbb{P}^n_{\overline{K}}$, the following hold.*

- *If V is an ideal-theoretic complete intersection, then it is a set-theoretic complete intersection.*
- *If V is a normal set-theoretic complete intersection, then it is absolutely irreducible.*
- *If V is nonsingular, then it is normal.*

2.2. **Multiprojective space.** Let $\mathbb{N} = \mathbb{Z}_{\geq 0}$ be the set of nonnegative integers, and let $\boldsymbol{n} = (n_1, \ldots, n_s) \in \mathbb{N}^s$. We define $|\boldsymbol{n}| = n_1 + \cdots + n_s$ and $\boldsymbol{n}! = n_1! \cdots n_s!$. Given $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^s$, we write $\boldsymbol{\alpha} \geq \boldsymbol{\beta}$ whenever $\alpha_i \geq \beta_i$ holds for $1 \leq i \leq s$. For $\boldsymbol{d} = (d_1, \ldots, d_s) \in \mathbb{N}^s$, the set $\mathbb{N}^{\boldsymbol{n}+\boldsymbol{1}}_{\boldsymbol{d}} = \mathbb{N}^{n_1+1}_{d_1} \times \cdots \times \mathbb{N}^{n_s+1}_{d_s}$ consists of the elements $\boldsymbol{a} = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_s) \in \mathbb{N}^{n_1+1} \times \cdots \times \mathbb{N}^{n_s+1}$ with $|\boldsymbol{a}_i| = d_i$ for $1 \leq i \leq s$.

We denote by $\mathbb{P}^{\boldsymbol{n}}_{\overline{K}}$ the multiprojective space $\mathbb{P}^{\boldsymbol{n}}_{\overline{K}} = \mathbb{P}^{n_1}_{\overline{K}} \times \cdots \times \mathbb{P}^{n_s}_{\overline{K}}$ defined over $\overline{K}$. For $1 \leq i \leq s$, let $X_i = \{X_{i,0}, \ldots, X_{i,n_i}\}$ be disjoint sets of $n_i + 1$ variables and let $\boldsymbol{X} = \{X_1, \dots, X_s\}$. A *multihomogeneous* polynomial $f \in K[\boldsymbol{X}]$ of multidegree $\boldsymbol{d} = (d_1, \dots, d_s)$ is a polynomial which is homogeneous of degree d_i in X_i for $1 \le i \le s$. An ideal $I \subset K[\boldsymbol{X}]$ is *multihomogeneous* if it is generated by a family of multihomogeneous polynomials. For any such ideal, we denote by $Z(I) \subset \mathbb{P}^{\boldsymbol{n}}_{\overline{K}}$ the variety defined (over K) as its set of common zeros. In particular, a hypersurface in $\mathbb{P}^{\boldsymbol{n}}_{\overline{K}}$ defined over K is the set of zeros of a multihomogeneous polynomial of $K[\boldsymbol{X}]$. The notions of irreducibility and dimension of a variety in $\mathbb{P}^{\boldsymbol{n}}_{\overline{K}}$ are defined as in the projective space.

2.2.1. *Mixed degrees.* We discuss the concept of *mixed degree* of a multiprojective variety and a few of its properties, following the exposition in D'Andrea *et al.* (2013). Let $V \subset \mathbb{P}^{\boldsymbol{n}}_{\overline{K}}$ be an irreducible variety defined over $\overline{K}$ of dimension m and let $I(V) \subset \overline{K}[\boldsymbol{X}]$ be its multihomogeneous ideal. The quotient ring $\overline{K}[\boldsymbol{X}]/I(V)$ is multigraded and its part of multidegree $\boldsymbol{b} \in \mathbb{N}^s$ is denoted by $(\overline{K}[\boldsymbol{X}]/I(V))_{\boldsymbol{b}}$. The *Hilbert–Samuel* function of V is the function $H_V : \mathbb{N}^s \to \mathbb{N}$ defined as $H_V(\boldsymbol{b}) = \dim(\overline{K}[\boldsymbol{X}]/I(V))_{\boldsymbol{b}}$. It turns out that there exist $\boldsymbol{\delta}_0 \in \mathbb{N}^s$ and a unique polynomial $P_V \in \mathbb{Q}[T_1, \dots, T_s]$ of degree m such that $P_V(\boldsymbol{\delta}) = H_V(\boldsymbol{\delta})$ for every $\boldsymbol{\delta} \in \mathbb{N}^s$ with $\boldsymbol{\delta} \ge \boldsymbol{\delta}_0$; see D'Andrea *et al.* (2013), Proposition 1.8. For $\boldsymbol{b} \in \mathbb{N}^s_{\le m}$, we define the *mixed degree of* V *of index* $\boldsymbol{b}$ as the nonnegative integer

$$\deg_{\boldsymbol{b}}(V) = \boldsymbol{b}! \cdot \mathrm{coeff}_{\boldsymbol{b}}(P_V).$$

This notion can be extended to equidimensional varieties and, more generally, to equidimensional cycles (formal integer linear combinations of subvarieties of equal dimension) by linearity.

The Chow ring of $\mathbb{P}^{\boldsymbol{n}}_{\overline{K}}$ is the graded ring

$$A^*(\mathbb{P}^{\boldsymbol{n}}_{\overline{K}}) = \mathbb{Z}[\theta_1, \dots, \theta_s]/(\theta_1^{n_1+1}, \dots, \theta_s^{n_s+1}),$$

where each θ_i denotes the class of the inverse image of a hyperplane of $\mathbb{P}^{n_i}_{\overline{K}}$ under the projection $\mathbb{P}^{\boldsymbol{n}}_{\overline{K}} \to \mathbb{P}^{n_i}_{\overline{K}}$. Given a variety $V \subset \mathbb{P}^{\boldsymbol{n}}_{\overline{K}}$ of pure dimension m, its class in the Chow ring is

$$[V] = \sum_{\boldsymbol{b}} \deg_{\boldsymbol{b}}(V) \theta_1^{n_1 - b_1} \cdots \theta_s^{n_s - b_s} \in A^*(\mathbb{P}^{\boldsymbol{n}}_{\overline{K}}),$$

where the sum is over all $\boldsymbol{b} \in \mathbb{N}^s_{\le m}$ with $\boldsymbol{b} \le \boldsymbol{n}$. This is an homogeneous element of degree $|\boldsymbol{n}| - m$. In particular, if $\mathcal{H} \subset \mathbb{P}^{\boldsymbol{n}}_{\overline{K}}$ is a hypersurface and $f \in \overline{K}[\boldsymbol{X}]$ is a polynomial of minimal degree defining $\mathcal{H}$, then

$$[\mathcal{H}] = \sum_{1 \le i \le s} \deg_{X_i}(f)\, \theta_i; \tag{2.3}$$

see D'Andrea *et al.* (2013), Proposition 1.10.

A fundamental tool for estimates of mixed degrees involving intersections of multiprojective varieties is the following multiprojective version of the Bézout theorem, called the *multihomogeneous Bézout theorem*;

see D'Andrea *et al.* (2013), Theorem 1.11. If $V \subset \mathbb{P}_{\overline{K}}^{\boldsymbol{n}}$ is a multiprojective variety of pure dimension $m > 0$ and $f \in \overline{K}[\boldsymbol{X}]$ is a multihomogeneous polynomial such that $V \cap Z(f)$ is of pure dimension $m - 1$, then

$$[V \cap Z(f)] = [V] \cdot [Z(f)]. \tag{2.4}$$

Finally, the following result shows that mixed degrees are monotonic with respect to linear projections. Let $\boldsymbol{l} = (l_1, \ldots, l_s) \in \mathbb{N}^s$ be an s–tuple with $\boldsymbol{l} \leq \boldsymbol{n}$ and let $\pi : \mathbb{P}_{\overline{K}}^{\boldsymbol{n}} \dashrightarrow \mathbb{P}_{\overline{K}}^{\boldsymbol{l}}$ be the linear projection which takes the first $l_i + 1$ coordinates of each coordinate x_i of each point $\boldsymbol{x} = (x_1, \ldots, x_s) \in \mathbb{P}_{\overline{K}}^{\boldsymbol{r}}$, namely,

$$\pi(x_{i,j} \colon 1 \leq i \leq s, 0 \leq j \leq n_i) = (x_{i,j} \colon 1 \leq i \leq s, 0 \leq j \leq l_i).$$

This rational map induces the following injective $\mathbb{Z}$–linear map:

$$\jmath : A^*(\mathbb{P}_{\overline{K}}^{\boldsymbol{l}}) \to A^*(\mathbb{P}_{\overline{K}}^{\boldsymbol{n}}), \quad \jmath(P) = \boldsymbol{\theta}^{\boldsymbol{n}-\boldsymbol{l}} P.$$

If $V \subset \mathbb{P}_{\overline{K}}^{\boldsymbol{n}}$ is a variety of pure dimension m and $\overline{\pi(V)}$ is also of pure dimension m, then

$$\jmath([\overline{\pi(V)}]) \leq [V]; \tag{2.5}$$

see D'Andrea *et al.* (2013), Proposition 1.16. Equivalently, $\deg_{\boldsymbol{b}}(\pi_* V) \leq \deg_{\boldsymbol{b}} V$ for any $\boldsymbol{b} \in \mathbb{N}^s_{\leq m}$, where $\pi_* V = \deg(\pi|_V) \overline{\pi(V)}$ and $\deg(\pi|_V) = [\overline{K}(V) : \overline{K}(\overline{\pi(V)})]$.

2.3. **Varieties over a finite field $\mathbb{F}_q$.** In the following, $\mathbb{P}_{\mathbb{F}}^n$ is the projective n–dimensional space over an algebraic closure $\mathbb{F}$ of $\mathbb{F}_q$, endowed with its Zariski topology. $\mathbb{P}^n(\mathbb{F}_q)$ is the n–dimensional projective space over $\mathbb{F}_q$, of cardinality

$$p_n = \#\mathbb{P}^n(\mathbb{F}_q) = q^n + q^{n-1} + \cdots + 1. \tag{2.6}$$

We denote by $V(\mathbb{F}_q)$ the set of $\mathbb{F}_q$–rational points of a projective variety $V \subset \mathbb{P}_{\mathbb{F}}^n$, namely, $V(\mathbb{F}_q) = V \cap \mathbb{P}^n(\mathbb{F}_q)$. If V is of dimension m and degree δ, we have

$$\#V(\mathbb{F}_q) \leq \delta \, p_m; \tag{2.7}$$

see Ghorpade & Lachaud (2002), Proposition 12.1, or Cafure & Matera (2007), Proposition 3.1. For $\boldsymbol{n} = (n_1, \ldots, n_s) \in \mathbb{N}^s_{\geq 1}$, $\mathbb{P}_{\mathbb{F}}^{\boldsymbol{n}} = \mathbb{P}_{\mathbb{F}}^{n_1} \times \cdots \times \mathbb{P}_{\mathbb{F}}^{n_s}$ is the multiprojective space over $\mathbb{F}$. Let $f \in \mathbb{F}[\boldsymbol{X}]$ be multihomogeneous of multidegree $\boldsymbol{d} = (d_1, \ldots, d_s)$. The following provides a highly useful upper bound on the number of $\mathbb{F}_q$–rational zeros of f in $\mathbb{P}^{\boldsymbol{n}}(\mathbb{F}_q)$, which generalizes (2.7) to the multiprojective setting.

For $\boldsymbol{\varepsilon} \in \mathbb{N}^s$ and $\boldsymbol{n} \geq \boldsymbol{\varepsilon}$, we use the notations $\boldsymbol{d}^{\boldsymbol{\varepsilon}} = d_1^{\varepsilon_1} \cdots d_s^{\varepsilon_s}$ and $p_{\boldsymbol{n}-\boldsymbol{\varepsilon}} = p_{n_1-\varepsilon_1} \cdots p_{n_s-\varepsilon_s}$.

Fact 2.2 (Cafure *et al.* (2015), Proposition 3.1)**.** *Let $f \in \mathbb{F}[\boldsymbol{X}]$ be a multihomogeneous polynomial of multidegree* $\boldsymbol{d}$ *with $d_i \le q$ for all i, and let N be the number of zeros of f in $\mathbb{P}^{\boldsymbol{n}}(\mathbb{F}_q)$. Then*

$$N \le \sum_{\boldsymbol{\varepsilon} \in \{0,1\}^s \setminus \{\boldsymbol{0}\}} (-1)^{|\boldsymbol{\varepsilon}|+1} \boldsymbol{d}^{\boldsymbol{\varepsilon}} p_{\boldsymbol{n}-\boldsymbol{\varepsilon}}.$$

2.4. **General setting.** It is convenient to fix the following notation:

$$\begin{aligned}
&\text{integers } n \text{ and } s \text{ with } 0 < s < n,\\
&\boldsymbol{d} = (d_1, \ldots, d_s) \in \mathbb{N}^s \text{ with } d_1 \ge d_2 \ge \cdots \ge d_s \ge 1 \text{ and } d_1 \ge 2,\\
&\delta = d_1 \cdots d_s,\\
&\sigma = (d_1 - 1) + \cdots + (d_s - 1),\\
&D_i = \binom{d_i + n}{n} - 1 \text{ for } 1 \le i \le s,\\
&\boldsymbol{D} = (D_1, \ldots, D_s) \in \mathbb{N}^s,\\
&|\boldsymbol{D}| = D_1 + \cdots + D_s.
\end{aligned} \tag{2.8}$$

Let K be a field. Each s–tuple $\boldsymbol{f} = (f_1, \ldots, f_s)$ with $f_i \in K[X] = K[X_0, \ldots, X_n]$ homogeneous of degree $\deg f_i = d_i$ is represented by a point in the multiprojective space $\mathbb{P}_{\overline{K}}^{\boldsymbol{D}} = \mathbb{P}_{\overline{K}}^{D_1} \times \cdots \times \mathbb{P}_{\overline{K}}^{D_s}$. More precisely, let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_s)$ be a point of $\mathbb{P}_{\overline{K}}^{\boldsymbol{D}}$. We label the D_i+1 coordinates of each λ_i by the D_i+1 multi-indices $\boldsymbol{\alpha} \in \mathbb{N}_{d_i}^{n+1}$, namely, $\lambda_i = (\lambda_{i,\boldsymbol{\alpha}} \colon |\boldsymbol{\alpha}| = d_i)$. Then we associate each point $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_s)$ with the s–tuple of polynomials $\boldsymbol{f} = (f_1, \ldots, f_s)$ defined as $f_i = \sum_{|\boldsymbol{\alpha}|=d_i} \lambda_{\boldsymbol{\alpha},i} X^{\boldsymbol{\alpha}}$ for $1 \le i \le s$. In the following, the symbol $\boldsymbol{f} = (f_1, \ldots, f_s)$ shall denote either an s–tuple of homogeneous polynomials of $K[X_0, \ldots, X_n]$ with degree pattern $(d_1, \ldots, d_s)$ or the corresponding point in $\mathbb{P}_{\overline{K}}^{\boldsymbol{D}}$.

Let $\{F_{i,\boldsymbol{\alpha}} : |\boldsymbol{\alpha}| = d_i\}$ be a set of $D_i + 1$ variables over $\overline{K}$ for $1 \le i \le s$. We shall consider the formal polynomial $F_i = \sum_{|\boldsymbol{\alpha}|=d_i} F_{i,\boldsymbol{\alpha}} X^{\boldsymbol{\alpha}}$, which is homogeneous of degree d_i in the variables $X_0, \ldots, X_n$. We use the notations $\mathrm{coeffs}(F_i) = \{F_{i,\boldsymbol{\alpha}} \colon |\boldsymbol{\alpha}| = d_i\}$ for $1 \le i \le s$ and $\mathrm{coeffs}(\boldsymbol{F}) = \cup_{1 \le i \le s} \mathrm{coeffs}(F_i)$. The coordinate ring of $\mathbb{P}_{\overline{K}}^{\boldsymbol{D}}$ is represented by the polynomial ring $\overline{K}[\mathrm{coeffs}(\boldsymbol{F})]$. The genericity polynomials P to be defined are elements of this ring, and given some polynomial sequence $\boldsymbol{f}$ as above, it is well-defined whether $P(\boldsymbol{f}) = 0$ or not.

2.4.1. *Multivariate resultants.* As we shall rely repeatedly on multivariate resultants in our arguments, we briefly recall their definition and basic properties. Given generic homogeneous polynomials $f_1, \ldots, f_{n+1} \in K[X_0, \ldots, X_n]$ of degrees $d_1, \ldots, d_{n+1}$, namely, given a generic point in the multiprojective space $\mathbb{P}_{\overline{K}}^{\boldsymbol{D}} = \mathbb{P}_{\overline{K}}^{D_1} \times \cdots \times \mathbb{P}_{\overline{K}}^{D_{n+1}}$, it is well-known that the projective variety $V(f_1, \ldots, f_{n+1}) \subset \mathbb{P}_{\overline{K}}^{n}$ they define is empty.

The multivariate resultant of formal polynomials $F_1, \ldots, F_{n+1}$ of degrees $d_1, \ldots, d_{n+1}$ is the unique irreducible multihomogeneous polynomial $\mathrm{Res} \in K[\mathrm{coeffs}(F_1), \ldots, \mathrm{coeffs}(F_{n+1})]$ with the following properties (see Cox *et al.* (1998), Chapter 3, Theorem 2.3):

- if $f_1, \ldots, f_{n+1} \in K[X_0, \ldots, X_n]$ are homogeneous polynomials of degrees $d_1, \ldots, d_{n+1}$, then $\mathrm{Res}(f_1, \ldots, f_{n+1}) = 0$ if and only if $V(f_1, \ldots, f_{n+1}) \subset \mathbb{P}^n_{\overline{K}}$ is nonempty.
- $\mathrm{Res}(X_0^{d_1}, \ldots, X_n^{d_{n+1}}) = 0$.

Further, Res has degree $d_1 \cdots d_{i-1} d_{i+1} \cdots d_{n+1}$ in the coefficients of F_i; see Cox *et al.* (1998), Chapter 3, Theorem 3.1.

3. Set-theoretic and ideal-theoretic complete intersections

In this section we obtain genericity polynomials for the set of s–tuples of homogeneous polynomials as above which do not define set-theoretic and ideal-theoretic complete intersections.

3.1. **Set-theoretic complete intersections.** We first consider the set of s–tuples $\boldsymbol{f}$ of homogeneous polynomials of $K[X_0, \ldots, X_n]$ with degree pattern $(d_1, \ldots, d_s)$ defining a set-theoretic complete intersection. For this purpose, we introduce the following incidence variety:

$$W = \{(\boldsymbol{f}, x) \in \mathbb{P}^{\boldsymbol{D}}_{\overline{K}} \times \mathbb{P}^n_{\overline{K}} : \boldsymbol{f}(x) = \boldsymbol{0}\}. \tag{3.1}$$

This incidence variety is well-known. For the sake of completeness, we establish here its most important geometric properties.

Lemma 3.1. *W is absolutely irreducible of dimension $|\boldsymbol{D}| + n - s$.*

Proof. Let $\phi : W \to \mathbb{P}^n_{\overline{K}}$ be the restriction of the projection $\mathbb{P}^{\boldsymbol{D}}_{\overline{K}} \times \mathbb{P}^n_{\overline{K}} \to \mathbb{P}^n_{\overline{K}}$ to the second argument. Then ϕ is a closed mapping, because it is the restriction to W of the projection $\mathbb{P}^{\boldsymbol{D}}_{\overline{K}} \times \mathbb{P}^n_{\overline{K}} \to \mathbb{P}^n_{\overline{K}}$, which is a closed mapping.

As W is a closed set of a multiprojective space, it is a projective variety. Furthermore, each fiber $\phi^{-1}(x)$ is a linear (irreducible) variety of dimension $|\boldsymbol{D}| - s > 0$, and $\phi : W \to \mathbb{P}^n_{\overline{K}}$ is surjective. Then Shafarevich (1994), §I.6.3, Theorem 8, shows that W is irreducible.

Finally, since W is defined by s polynomials which form a regular sequence of $K[\mathrm{coeffs}(\boldsymbol{F}), X]$, we see that $\dim W = |\boldsymbol{D}| + n - s$. □

By the theorem on the dimension of fibers, a generic $\boldsymbol{f}$ as above defines a projective variety $Z(\boldsymbol{f}) \subset \mathbb{P}^n_{\overline{K}}$ of dimension $n - s$, which is necessarily a set-theoretic complete intersection. Our next result provides quantitative information concerning such $\boldsymbol{f}$.

Theorem 3.2. *In the notation (2.8), there exists a nonzero multihomogeneous polynomial $P_{\mathrm{stci}} \in K[\mathrm{coeffs}(\boldsymbol{F})]$, of degree at most δ/d_i in* *each set of variables* coeffs(F_i) *for* $1 \leq i \leq s$, *with the following property: for any* $\boldsymbol{f} \in \mathbb{P}_{\overline{K}}^{\boldsymbol{D}}$ *with* $P_{\mathrm{stci}}(\boldsymbol{f}) \neq 0$, *the variety* $Z(\boldsymbol{f})$ *has dimension* $n-s$. *In particular,* $Z(\boldsymbol{f})$ *is a set-theoretic complete intersection and* $\boldsymbol{f}$ *is a regular sequence.*

Proof. Let $\boldsymbol{f} = (f_1, \dots, f_s)$ be an arbitrary point of $\mathbb{P}_{\overline{K}}^{\boldsymbol{D}}$. Suppose that the variety $Z(\boldsymbol{f}) \subset \mathbb{P}_{\overline{K}}^n$ has dimension $\dim Z(\boldsymbol{f}) > n - s$. Then $Z(\boldsymbol{f}, X_s, \dots, X_n)$ is not empty. It follows that the set of s–tuples of polynomials $\boldsymbol{f}$ defining a variety of dimension strictly greater than $n-s$ is contained in the set of $\boldsymbol{f}$ such that $Z(\boldsymbol{f}, X_s, \dots, X_n)$ is not empty.

The multivariate resultant of formal polynomials $F_1, \dots, F_s$, $F_{s+1}, \dots, F_{n+1}$ of degrees $d_1, \dots, d_s, 1, \dots, 1$ is an irreducible multihomogeneous polynomial of $K[\mathrm{coeffs}(F_1), \dots, \mathrm{coeffs}(F_{n+1})]$ of degree $d_1 \cdots d_{i-1} d_{i+1} \cdots d_{n+1}$ in the coefficients of F_i. In particular, the multivariate resultant of $F_1, \dots, F_s, X_s, \dots, X_n$ is a nonzero multihomogeneous polynomial $P_{\mathrm{stci}} \in K[\mathrm{coeffs}(\boldsymbol{F})]$ of degree δ/d_i in the coefficients of F_i for $1 \leq i \leq s$.

We claim that the multihomogeneous polynomial P_{stci} satisfies the requirements of the theorem. Indeed, let $\boldsymbol{f} \in \mathbb{P}_{\overline{K}}^{\boldsymbol{D}}$ with $P_{\mathrm{stci}}(\boldsymbol{f}) \neq 0$. Then the multivariate resultant of $\boldsymbol{f}, X_s, \dots, X_n$ does not vanish, and thus the projective variety $Z(\boldsymbol{f}, X_s, \dots, X_n) \subset \mathbb{P}_{\overline{K}}^n$ is empty, which implies that $Z(\boldsymbol{f})$ has dimension at most $n-s$. On the other hand, each irreducible component of $Z(\boldsymbol{f})$ has dimension at least $n-s$. We deduce that $Z(\boldsymbol{f})$ is of pure dimension $n-s$. Furthermore, as $Z(\boldsymbol{f})$ is defined by s homogeneous polynomials, we conclude that it is a set-theoretic complete intersection. This finishes the proof of the theorem. □

3.2. **Ideal-theoretic complete intersections.** Now we consider the set of s–tuples of homogeneous polynomials $\boldsymbol{f}$ as above defining a complete intersection.

For this purpose, we introduce another incidence variety:

$$W_{\mathrm{ci}} = \{(\boldsymbol{f}, x) \in \mathbb{P}_{\overline{K}}^{\boldsymbol{D}} \times \mathbb{P}_{\overline{K}}^n : \boldsymbol{f}(x) = \boldsymbol{0}, J(\boldsymbol{f})(x) = 0\}, \tag{3.2}$$

where $J(\boldsymbol{f}) = \det(\partial f_i / \partial X_j : 1 \leq i, j \leq s)$ is the Jacobian determinant of $\boldsymbol{f}$ with respect to $X_1, \dots, X_s$.

Lemma 3.3. W_{ci} *is of pure dimension* $|\boldsymbol{D}| + n - s - 1$.

Proof. We have $W_{\mathrm{ci}} = W \cap \{J(\boldsymbol{f})(x) = 0\}$, where W is the incidence variety of (3.1). We claim that $J(\boldsymbol{f})(x)$ does not vanish identically on W. Indeed, fix a squarefree polynomial $f_i \in \overline{K}[T]$ of degree d_i for $1 \leq i \leq s$ and let $f_i^h \in \overline{K}[X_0, X_i]$ be the homogenization of $f_i(X_i)$ with homogenizing variable X_0. Denote

$$\boldsymbol{f}_0 = (f_1^h(X_0, X_1), \dots, f_s^h(X_0, X_s)). \tag{3.3}$$

Then $\{\boldsymbol{f}_0\} \times Z(\boldsymbol{f}_0)$ is contained in W and $J(\boldsymbol{f}_0)$ does not vanish identically on $Z(\boldsymbol{f}_0)$, which shows the claim.

According to Lemma 3.1, W is absolutely irreducible of dimension $|\boldsymbol{D}|+n-s$. Therefore, by the claim we see that $W_{\mathrm{ci}} = W \cap \{J(\boldsymbol{f})(x) = 0\}$ is of pure dimension $|\boldsymbol{D}| + n - s - 1$. □

We denote by $\pi\colon W_{\mathrm{ci}} \to \mathbb{P}_{\overline{K}}^{\boldsymbol{D}}$ the projection to the first argument and have the following result.

Lemma 3.4. *$\pi\colon W_{\mathrm{ci}} \to \mathbb{P}_{\overline{K}}^{\boldsymbol{D}}$ is a dominant mapping.*

Proof. Let $\boldsymbol{f}_0$ be the s–tuple of polynomials of (3.3). Then the fiber $\pi^{-1}(\boldsymbol{f}_0)$ has dimension $n - s - 1$. Let $\mathcal{C}$ be an irreducible component of W_{ci} such that $\boldsymbol{f}_0 \in \pi(\mathcal{C})$. It is clear that $\dim \pi(\mathcal{C}) \leq |\boldsymbol{D}|$, and $\dim \mathcal{C} = |\boldsymbol{D}| + n - s - 1$ by Lemma 3.3. The theorem on the dimension of fibers shows that

$$\dim \mathcal{C} - \dim \pi(\mathcal{C}) = |\boldsymbol{D}| + n - s - 1 - \dim \pi(\mathcal{C}) \leq \dim \pi^{-1}(\boldsymbol{f}_0) = n - s - 1.$$

Thus $\dim \pi(\mathcal{C}) \geq |\boldsymbol{D}|$ and hence $\dim \pi(\mathcal{C}) = |\boldsymbol{D}|$. It follows that $\pi(W_{\mathrm{ci}}) = \mathbb{P}_{\overline{K}}^{\boldsymbol{D}}$. □

A consequence of Lemma 3.4 is that a generic fiber $\pi^{-1}(\boldsymbol{f})$ has dimension $n - s - 1$. In particular, for such an $\boldsymbol{f}$ the variety $Z(\boldsymbol{f})$ is of pure dimension $n - s$. Thus, $\boldsymbol{f}$ is a regular sequence of $\overline{K}[X]$ and the hypersurface defined by the Jacobian determinant $J(\boldsymbol{f})$ intersects $Z(\boldsymbol{f})$ in a subvariety of $Z(\boldsymbol{f})$ of dimension $n - s - 1$. As we show below, this implies that $\boldsymbol{f}$ defines a radical ideal and $Z(\boldsymbol{f})$ is a complete intersection.

We now turn this into quantitative information on a genericity polynomial whose set of zeros contains all systems not defining a complete intersection.

Theorem 3.5. *In the notation (2.8), there exists a nonzero multihomogeneous polynomial $P_{\mathrm{ci}} \in K[\mathrm{coeffs}(\boldsymbol{F})]$ with*

$$\deg_{\mathrm{coeffs}(F_i)} P_{\mathrm{ci}} \leq \delta(\frac{\sigma}{d_i} + 1) \leq 2\sigma\delta$$

for $1 \leq i \leq s$ such that any $\boldsymbol{f} = (f_1, \ldots, f_s) \in \mathbb{P}_{\overline{K}}^{\boldsymbol{D}}$ with $P_{\mathrm{ci}}(\boldsymbol{f}) \neq 0$ satisfies the following properties:

- *$f_1, \ldots, f_s$ form a regular sequence of $\overline{K}[X_0, \ldots, X_n]$,*
- *the ideal of $\overline{K}[X_0, \ldots, X_n]$ generated by $f_1, \ldots, f_s$ is radical.,*
- *$Z(\boldsymbol{f})$ is an ideal-theoretic complete intersection of dimension $n - s$ and degree δ.*

Proof. Let $\boldsymbol{f} = (f_1, \ldots, f_s)$ be a point of $\mathbb{P}_{\overline{K}}^{\boldsymbol{D}}$. If $Z\big(\boldsymbol{f}, J(\boldsymbol{f})\big) \subset \mathbb{P}_{\overline{K}}^{n}$ has dimension strictly greater than $n-s-1$, then $Z(\boldsymbol{f}, J(\boldsymbol{f}), X_{s+1}, \ldots, X_n)$ is not empty. We conclude that the set of $\boldsymbol{f}$ with $\dim Z\big(\boldsymbol{f}, J(\boldsymbol{f})\big) > n-s-1$ is contained in the set of $\boldsymbol{f}$ for which $Z(\boldsymbol{f}, J(\boldsymbol{f}), X_{s+1}, \ldots, X_n)$ is not empty.

Let $\boldsymbol{f}$ be a point of $\mathbb{P}_{\overline{K}}^{\boldsymbol{D}}$ such that $Z(\boldsymbol{f}, J(\boldsymbol{f}), X_{s+1}, \ldots, X_n)$ is not empty. Then the resultant of $\boldsymbol{f}, J(\boldsymbol{f}), X_{s+1}, \ldots, X_n$ must vanish. The multivariate resultant of $F_1,\dots,F_s,J(\boldsymbol{F}),X_{s+1},\dots,X_n$ is a nonzero polynomial $P_{\mathrm{ci}}\in K[\mathrm{coeffs}(\boldsymbol{F})]$. Indeed, let $\boldsymbol{f}_0\in\mathbb{P}^{\boldsymbol{D}}_{\overline{K}}$ be the point defined in (3.3). Then it is easy to see that $Z(\boldsymbol{f}_0,J(\boldsymbol{f}_0),X_{s+1},\dots,X_n)$ is empty, which implies that $P_{\mathrm{ci}}(\boldsymbol{f}_0)\neq 0$.

We claim that the multihomogeneous polynomial $P_{\mathrm{ci}}\in K[\mathrm{coeffs}(\boldsymbol{F})]$ satisfies the requirements of the theorem.

In order to show this claim, let $\boldsymbol{f}\in\mathbb{P}^{\boldsymbol{D}}_{\overline{K}}$ with $P_{\mathrm{ci}}(\boldsymbol{f})\neq 0$. Then the variety $Z(\boldsymbol{f},J(\boldsymbol{f}),X_{s+1},\dots,X_n)$ is empty, which implies that $V'=Z(\boldsymbol{f},J(\boldsymbol{f}))$ has dimension at most $n-s-1$. On the other hand, each irreducible component of V' has dimension at least $n-s-1$ by definition. We conclude that V' is of pure dimension $n-s-1$.

Furthermore, as each irreducible component of $V=Z(\boldsymbol{f})$ has dimension at least $n-s$, it follows that $\dim V\cap Z(J(\boldsymbol{f}))\geq \dim V-1=n-s-1$, with equality if and only if V is of pure dimension $n-s$ and $Z(J(\boldsymbol{f}))$ cuts properly each irreducible component of V. But $V'=V\cap Z(J(\boldsymbol{f}))$ has dimension $n-s-1$ by the previous argument, which proves that V is of pure dimension $n-s$ and $Z(J(\boldsymbol{f}))$ cuts properly each irreducible component of V, namely the ideal defined by $J(\boldsymbol{f})$ has codimension 1 in $K[V]$. Thus the ideal generated for all the $s\times s$-minors of $J(\boldsymbol{f})$ has codimension at least 1 in $K[V]$. We conclude that $f_1,\dots,f_s$ form a regular sequence of $\overline{K}[X]$ and Eisenbud (1995), Theorem 18.15, proves that $f_1,\dots,f_s$ generate a radical ideal of $\overline{K}[X]$, so that V is a complete intersection.

For an upper bound on the degree of P_{ci}, we recall the multivariate resultant of formal homogeneous polynomials $F_1,\dots,F_s,F_{s+1},\dots,F_{n+1}$ of degrees $d_1,\dots,d_s,\sigma,1,\dots,1$ is a multihomogeneous element of $K[\mathrm{coeffs}(F_1),\dots,\mathrm{coeffs}(F_{n+1})]$ of degree $d_1\cdots d_{i-1}d_{i+1}\cdots d_s\sigma=\sigma\delta/d_i$ in the coefficients of F_i for $1\leq i\leq s$ and degree δ in the coefficients of F_{s+1}. We deduce that $P_{\mathrm{ci}}\in K[\mathrm{coeffs}(\boldsymbol{F})]$ has degree $\sigma\delta/d_i+\delta$ in the variables $\mathrm{coeffs}(F_i)$ for $1\leq i\leq s$.

The third property follows from the Bézout theorem (2.2). □

3.2.1. *Complete intersections defined over* $\mathbb{F}_q$. From the theorem, we now derive a bound over a finite field $\mathbb{F}_q$. The number of all s–tuples of homogeneous polynomials of $\mathbb{F}_q[X_0,\dots,X_n]$ with degree sequence $\boldsymbol{d}$ is

$$\#\mathbb{P}^{\boldsymbol{D}}(\mathbb{F}_q)=p_{\boldsymbol{D}}=\prod_{1\leq i\leq s}p_{D_i}. \tag{3.4}$$

We first present a general lower bound on the number of nonzeros of a multihomogeneous polynomial with bounded degrees.

Proposition 3.6. *Let* $P\in K[\mathrm{coeffs}(\boldsymbol{F})]$ *be a multihomogeneous polynomial with* $\deg_{\mathrm{coeffs}(F_i)}(P)\leq e_i\leq e\leq q$ *for* $1\leq i\leq s$*, and let* N *be* *the number of* $\boldsymbol{f} \in \mathbb{P}^{\boldsymbol{D}}(\mathbb{F}_q)$ *with* $P(\boldsymbol{f}) \neq 0$. *Then*

$$1 - \frac{se}{q} \leq \prod_{1 \leq i \leq s} \bigl(1 - \frac{e_i}{q}\bigr) \leq \frac{N}{p_{\boldsymbol{D}}} \leq 1.$$

The leftmost inequality assumes additionally that $q \geq es/3$.

Proof. The upper bound being obvious, we prove the lower bound. As $q \geq e_i$ for all i, Fact 2.2 shows that

$$\#\{\boldsymbol{f} \in \mathbb{P}^{\boldsymbol{D}}(\mathbb{F}_q) : P(\boldsymbol{f}) = 0\} \leq \sum_{\boldsymbol{\varepsilon} \in \{0,1\}^s \setminus \{\boldsymbol{0}\}} (-1)^{|\boldsymbol{\varepsilon}|+1} \boldsymbol{e}^{\boldsymbol{\varepsilon}} p_{\boldsymbol{D}-\boldsymbol{\varepsilon}},$$

where $\boldsymbol{e} = (e_1, \ldots, e_s)$. Using the inequality

$$\frac{p_{D_i} - e_i\, p_{D_i - 1}}{p_{D_i}} \geq 1 - \frac{e_i}{q} \geq 1 - \frac{e}{q}$$

for $1 \leq i \leq s$, we conclude that

$$\begin{aligned} N = \#\{\boldsymbol{f} \in \mathbb{P}^{\boldsymbol{D}}(\mathbb{F}_q) : P(\boldsymbol{f}) \neq 0\} &\geq p_{\boldsymbol{D}} - \sum_{\boldsymbol{\varepsilon} \in \{0,1\}^s \setminus \{\boldsymbol{0}\}} (-1)^{|\boldsymbol{\varepsilon}|+1} \boldsymbol{e}^{\boldsymbol{\varepsilon}} p_{\boldsymbol{D}-\boldsymbol{\varepsilon}} \\ &= \sum_{\boldsymbol{\varepsilon} \in \{0,1\}^s} (-1)^{|\boldsymbol{\varepsilon}|} \boldsymbol{e}^{\boldsymbol{\varepsilon}} p_{\boldsymbol{D}-\boldsymbol{\varepsilon}} = \prod_{1 \leq i \leq s} \bigl(p_{D_i} - e_i\, p_{D_i - 1}\bigr) \\ &\geq p_{\boldsymbol{D}} \prod_{1 \leq i \leq s} \bigl(1 - \frac{e_i}{q}\bigr) \geq p_{\boldsymbol{D}} \bigl(1 - \frac{e}{q}\bigr)^s \geq p_{\boldsymbol{D}} \bigl(1 - \frac{se}{q}\bigr). \end{aligned}$$

The last inequality assumes $q \geq es/3$, so that in the binomial expansion of the sth power, each positive even term (after the first two) is at least as large as the following negative odd one. □

An important feature is the fact that the numerator in the lower bound depends on the geometric system parameters s, e_i, and e, but not on q. This will be applied in several scenarios. We then only state the concise leftmost lower bound. The reader can easily substitute the more precise product lower bound if required, also allowing a slightly relaxed lower bound on q. Furthermore, there exist polynomial systems not having the desired property, for example $f_i = x_1^{d_i}$ for all i, so that we may replace the upper bound $N/p_{\boldsymbol{D}} \leq 1$ by $N/p_{\boldsymbol{D}} < 1$; this also holds for the other properties considered in this paper.

Combining Theorem 3.5 and Proposition 3.6, we obtain the following result.

Corollary 3.7. *In the notation (2.8), suppose that* $q \geq 2s\delta\sigma/3$. *Let* N_{ci} *be the number of* $\boldsymbol{f} \in \mathbb{P}^{\boldsymbol{D}}(\mathbb{F}_q)$ *defining a complete intersection* $Z(\boldsymbol{f}) \subset \mathbb{P}^n_{\mathbb{F}}$ *of dimension* $n - s$ *and degree* $\delta = d_1 \cdots d_s$. *Then*

$$1 - \frac{2s\delta\sigma}{q} \leq \frac{N_{\mathrm{ci}}}{p_{\boldsymbol{D}}} < 1.$$

When the geometric data are fixed, the lower bound increases with growing q. The corollary can also be interpreted as bounding the probability that a uniformly random $\boldsymbol{f} \in \mathbb{P}^{\boldsymbol{D}}(\mathbb{F}_q)$ defines a complete intersection $Z(\boldsymbol{f}) \subset \mathbb{P}^n_{\mathbb{F}}$ of dimension $n-s$ and degree $\delta = d_1 \cdots d_s$.

4. Absolutely irreducible and smooth complete intersections

Now we return to the general framework of the previous section, that is, we fix an arbitrary field K and consider a sequence $\boldsymbol{f} = (f_1, \ldots, f_s)$ of s homogeneous polynomials $f_1, \ldots, f_s \in K[X] = K[X_0, \ldots, X_n]$ with a given degree pattern $(d_1, \ldots, d_s)$. In the previous section we have shown that for a generic $\boldsymbol{f}$, the projective variety $Z(\boldsymbol{f}) \subset \mathbb{P}^n_{\overline{K}}$ is a complete intersection of dimension $n-s$ and degree $\delta = d_1 \cdots d_s$.

In this section we show that $Z(\boldsymbol{f})$ is absolutely irreducible and smooth for a generic $\boldsymbol{f}$, and more precisely that the $\boldsymbol{f}$ without this property are contained in a hypersurface whose degree we control.

4.1. **Smooth complete intersections.** First we analyze smoothness. For this purpose, we introduce a further incidence variety. Let $\mathcal{M}_{\boldsymbol{F}} = (\partial F_i / \partial X_j : 1 \leq i \leq s, 0 \leq j \leq n)$ denote the Jacobian matrix of the formal homogeneous polynomials $F_1, \ldots, F_s$ of degrees $d_1, \ldots, d_s$. For $s+1 \leq k \leq n+1$, consider the $s \times s$–submatrix of $\mathcal{M}_{\boldsymbol{F}}$ consisting of the columns numbered $1, \ldots, s-1$ and $k-1$, and let $J_k(\boldsymbol{F}, X)$ be the corresponding determinant, namely,

$$J_k(\boldsymbol{F}, X) = \det\big(\partial F_i/\partial X_j \colon 1 \leq i \leq s, \quad j \in \{1,, \ldots, s-1, k-1\}\big). \tag{4.1}$$

We consider the incidence variety

$$W_{\text{nons}} = \{(\boldsymbol{f}, x) \in \mathbb{P}^{\boldsymbol{D}}_{\overline{K}} \times \mathbb{P}^n_{\overline{K}} \colon \boldsymbol{f}(x) = \boldsymbol{0}, J_k(\boldsymbol{f}, x) = 0 \text{ for } s+1 \leq k \leq n+1\}, \tag{4.2}$$

and have the following result.

Lemma 4.1. *The polynomials $J_{s+1}(\boldsymbol{F}, X), \ldots, J_{n+1}(\boldsymbol{F}, X), F_1, \ldots, F_s$ form a regular sequence of $K[\text{coeffs}(\boldsymbol{F}), X]$.*

Proof. For $s+1 \leq k \leq n+1$, let $\boldsymbol{\alpha}_k = (d_1-1, 0, \ldots, 1, 0, \ldots, 0)$ be the exponent of the monomial $X_0^{d_1-1} X_{k-1}$. The choice of $\boldsymbol{\alpha}_k$ implies that the nonzero monomial $F_{1,\boldsymbol{\alpha}_k} X_0^{d_1-1}$ occurs with nonzero coefficient in the representation of $\partial F_1 / \partial X_{k-1}$ as a sum of monomials. Furthermore, the Jacobian determinant $J_k(\boldsymbol{F}, X)$ is a primitive polynomial of $K[\text{coeffs}(\boldsymbol{F}) \setminus \{F_{1,\boldsymbol{\alpha}_k}\}, X][F_{1,\boldsymbol{\alpha}_k}]$ of degree 1 in $F_{1,\boldsymbol{\alpha}_k}$. In particular, $J_k(\boldsymbol{F}, X)$ is an irreducible element of $\overline{K}[\text{coeffs}(\boldsymbol{F}), X]$. On the other hand, if $l \neq k$, then $J_l(\boldsymbol{F}, X)$ has degree zero in $F_{1,\boldsymbol{\alpha}_k}$, since none of the entries of the matrix defining $J_l(\boldsymbol{F}, X)$ includes a derivative with respect to X_0 or X_{k-1}.

Since the multiprojective variety defined by $J_1(\boldsymbol{F}, X), \ldots, J_{k-1}(\boldsymbol{F}, X)$ is a "cylinder" in the direction corresponding to $F_{1,\boldsymbol{\alpha}_k}$ and $J_k(\boldsymbol{F}, X)$ is an irreducible nonconstant element of $K[\mathrm{coeffs}(\boldsymbol{F}) \setminus \{F_{1,\boldsymbol{\alpha}_k}\}, X][F_{1,\boldsymbol{\alpha}_k}]$, we conclude that $J_k(\boldsymbol{F}, X)$ is not a zero divisor modulo $J_1(\boldsymbol{F}, X), \ldots, J_{k-1}(\boldsymbol{F}, X)$ for $s + 1 \leq k \leq n + 1$.

Now, denote $\boldsymbol{\gamma}_i = (d_i, 0, \ldots, 0)$ for $1 \leq i \leq s$. Observe that no $J_k(\boldsymbol{F}, X)$ depends on any of the indeterminates $F_{i,\boldsymbol{\gamma}_i}$ for $1 \leq i \leq s$, since the partial derivatives of $F_1, \ldots, F_s$ with respect to X_0 are not included in any of the $s \times s$–submatrices of the Jacobian matrix $\mathcal{M}_{\boldsymbol{F}}$ defining the polynomials $J_k(\boldsymbol{F}, X)$. We conclude that each F_i is not a zero divisor modulo $J_{s+1}(\boldsymbol{F}, X), \ldots, J_{n+1}(\boldsymbol{F}, X), F_1, \ldots, F_{i-1}$. This finishes the proof of the lemma. □

Now we show that for a generic s–tuple $\boldsymbol{f}$ as above, the corresponding system defines a smooth complete intersection. We provide estimates on the degree of a hypersurface of $\mathbb{P}_{\overline{K}}^{\boldsymbol{D}}$ containing the elements $\boldsymbol{f}$ for which $Z(\boldsymbol{f})$ is not smooth.

Theorem 4.2. *In the notation (2.8), there exists a nonzero multihomogeneous polynomial $P_{\mathrm{nons}} \in K[\boldsymbol{F}]$ with*

$$\deg_{\mathrm{coeffs}(F_i)} P_{\mathrm{nons}} \leq \sigma^{n-s}\delta\Big(\frac{\sigma}{d_i} + n - s + 1\Big) \leq (\sigma + n)\sigma^{n-s}\delta$$

for $1 \leq i \leq s$ and such that for any $\boldsymbol{f} \in \mathbb{P}_{\overline{K}}^{\boldsymbol{D}}$ with $P_{\mathrm{nons}}(\boldsymbol{f}) \neq 0$, the variety $Z(\boldsymbol{f}) \subset \mathbb{P}_{\overline{K}}^{n}$ is a nonsingular complete intersection of dimension $n - s$ and degree δ.

Proof. From Lemma 4.1 we conclude that the incidence variety W_{nons} is of pure dimension $|\boldsymbol{D}| - 1$. Let $\pi : \mathbb{P}_{\overline{K}}^{\boldsymbol{D}} \times \mathbb{P}_{\overline{K}}^{n} \to \mathbb{P}_{\overline{K}}^{\boldsymbol{D}}$ be the projection to the first argument. Since π is a closed mapping, it follows that $\pi(W_{\mathrm{nons}})$ is a closed subset of $\mathbb{P}_{\overline{K}}^{\boldsymbol{D}}$ of dimension at most $|\boldsymbol{D}| - 1$. In particular, there exists $\boldsymbol{f} \in \mathbb{P}_{\overline{K}}^{\boldsymbol{D}}$ not belonging to $\pi(W_{\mathrm{nons}})$, which means that the equations $\{\boldsymbol{f}(x) = \boldsymbol{0}, J_{s+1}(\boldsymbol{f})(x) = 0, \ldots, J_{n+1}(\boldsymbol{f})(x) = 0\}$ define the empty set.

Let $D_{s+1} = \cdots = D_{n+1} = \binom{\sigma+n}{n} - 1$, let $\boldsymbol{D}' = (D_1, \ldots, D_{n+1})$ and $\mathbb{P}_{\overline{K}}^{\boldsymbol{D}'} = \mathbb{P}_{\overline{K}}^{\boldsymbol{D}} \times \mathbb{P}_{\overline{K}}^{D_{s+1}} \times \cdots \times \mathbb{P}_{\overline{K}}^{D_{n+1}}$. Let

$$K[\mathrm{coeffs}(\boldsymbol{F}')] = K[\mathrm{coeffs}(\boldsymbol{F}), \mathrm{coeffs}(F_{s+1}), \ldots, \mathrm{coeffs}(F_{n+1})]$$

and let $P \in K[\mathrm{coeffs}(\boldsymbol{F}')]$ be the multivariate resultant of formal polynomials $F_1, \ldots, F_{n+1}$ of degrees $d_1, \ldots, d_s, \sigma, \ldots, \sigma$. Denote by $\mathcal{H} \subset \mathbb{P}_{\overline{K}}^{\boldsymbol{D}'}$ the hypersurface defined by P. For any $\boldsymbol{f} \in \mathbb{P}_{\overline{K}}^{\boldsymbol{D}}$ we have $\boldsymbol{f} \in \pi(W_{\mathrm{nons}})$ if and only if the $(n+1)$–tuple $\big(\boldsymbol{f}, J_{s+1}(\boldsymbol{f}), \ldots, J_{n+1}(\boldsymbol{f})\big)$ belongs to $\mathcal{H}$. Let $\phi : \mathbb{P}_{\overline{K}}^{\boldsymbol{D}} \to \mathbb{P}_{\overline{K}}^{\boldsymbol{D}'}$ be the regular mapping defined as $\phi(\boldsymbol{f}) = \big(\boldsymbol{f}, J_{s+1}(\boldsymbol{f}), \ldots, J_{n+1}(\boldsymbol{f})\big)$. Then $\pi(W_{\mathrm{nons}})$ is the hypersurface of $\mathbb{P}_{\overline{K}}^{\boldsymbol{D}}$ defined by the polynomial $\phi^*(P)$, where $\phi^* : K[\mathrm{coeffs}(\boldsymbol{F}')] \to K[\mathrm{coeffs}(\boldsymbol{F})]$ is the K-algebra homomorphism defined by ϕ. Let $P_{\mathrm{nons}} \in$

$K[\mathrm{coeffs}(\boldsymbol{F})]$ be any polynomial of minimal degree defining $\pi(W_{\mathrm{nons}})$. We claim that P_{nons} satisfies the requirements of the theorem.

To this end, we estimate the multidegree of $\pi(W_{\mathrm{nons}})$. For this purpose, we consider the class $[W_{\mathrm{nons}}]$ of W_{nons} in the Chow ring $\mathcal{A}^*(\mathbb{P}_{\overline{K}}^{\boldsymbol{D}} \times \mathbb{P}_{\overline{K}}^{n})$ of $\mathbb{P}_{\overline{K}}^{\boldsymbol{D}} \times \mathbb{P}_{\overline{K}}^{n}$. We denote by θ_i the class of the inverse image of a hyperplane of $\mathbb{P}_{\overline{K}}^{D_i}$ under the ith canonical projection $\mathbb{P}_{\overline{K}}^{\boldsymbol{D}} \times \mathbb{P}_{\overline{K}}^{n} \to \mathbb{P}_{\overline{K}}^{D_i}$ for $1 \le i \le s$ and by θ_0 the class of the inverse image of a hyperplane of $\mathbb{P}_{\overline{K}}^{n}$ under the projection $\mathbb{P}_{\overline{K}}^{\boldsymbol{D}} \times \mathbb{P}_{\overline{K}}^{n} \to \mathbb{P}_{\overline{K}}^{n}$ to the second argument. By the definition (4.2) of W_{nons} and the multihomogeneous Bézout theorem (2.4), we obtain

$$\begin{aligned}[W_{\mathrm{nons}}] &= \Big(\prod_{1\le i\le s}(d_i\theta_0+\theta_i)\Big)(\sigma\theta_0+\theta_1+\cdots+\theta_s)^{n-s+1}\\ &= \sigma^{n-s+1}\delta\,\theta_0^{n+1} + \sigma^{n-s}\delta\sum_{1\le i\le s}\big(\frac{\sigma}{d_i}+n-s+1\big)\theta_0^n\theta_i + \mathcal{O}(\theta_0^{n-1}),\end{aligned}$$

where $\mathcal{O}\big(\theta_0^{n-1}\big)$ is a sum of terms of degree at most $n-1$ in θ_0.

On the other hand, by definition $[\pi(W_{\mathrm{nons}})] = \deg_{\mathrm{coeffs}(F_1)} P_{\mathrm{nons}}\,\theta_1 + \cdots + \deg_{\mathrm{coeffs}(F_s)} P_{\mathrm{nons}}\,\theta_s$. Let $\jmath : \mathcal{A}^*\big(\mathbb{P}_{\overline{K}}^{\boldsymbol{D}}\big) \hookrightarrow \mathcal{A}^*\big(\mathbb{P}_{\overline{K}}^{\boldsymbol{D}} \times \mathbb{P}_{\overline{K}}^{n}\big)$ be the injective $\mathbb{Z}$–map $Q \mapsto \theta_0^n Q$ induced by π. Then (2.5) shows that $\jmath([\pi(W_{\mathrm{nons}})]) \le [W_{\mathrm{nons}}]$, namely,

$$\jmath([\pi(W_{\mathrm{nons}})]) = \sum_{1\le i\le s}\deg_{\mathrm{coeffs}(F_i)} P_{\mathrm{nons}}\,\theta_0^n\theta_i \le [W_{\mathrm{nons}}],$$

where the inequality is understood in a coefficient–wise sense. This implies

$$\deg_{\mathrm{coeffs}(F_i)} P_{\mathrm{nons}} \le \sigma^{n-s}\delta\big(\frac{\sigma}{d_i}+n-s+1\big) \tag{4.3}$$

for $1 \le i \le s$. This proves the degree estimate in the statement of the theorem.

Further, let $\boldsymbol{f} = (f_1,\ldots,f_s) \in \mathbb{P}_{\overline{K}}^{\boldsymbol{D}}$ with $P_{\mathrm{nons}}(\boldsymbol{f}) \ne 0$. Then $\{\boldsymbol{f} = 0, J_{s+1}(\boldsymbol{f}) = 0,\ldots,J_{n+1}(\boldsymbol{f}) = 0\}$ is the empty projective subvariety of $\mathbb{P}_{\overline{K}}^{n}$. This implies that $Z(\boldsymbol{f})$ has dimension $n-s$ and $f_1,\ldots,f_s$ form a regular sequence of $\overline{K}[X]$. Furthermore, Eisenbud (1995), Theorem 18.15, proves that $f_1,\ldots,f_s$ generate a radical ideal of $\overline{K}[X]$. In particular, the singular locus of $Z(\boldsymbol{f})$ is contained in $\{\boldsymbol{f} = 0, J_{s+1}(\boldsymbol{f}) = 0,\ldots,J_{n+1}(\boldsymbol{f}) = 0\}$, which is an empty variety, showing thus that $Z(\boldsymbol{f})$ is a smooth variety. □

Let $\boldsymbol{f} \in \mathbb{P}_{\overline{K}}^{\boldsymbol{D}}$ with $P_{\mathrm{nons}}(\boldsymbol{f}) \ne 0$. Then $Z(\boldsymbol{f}) \subset \mathbb{P}_{\overline{K}}^{n}$ is a nonsingular complete intersection which, according to Fact 2.1, is absolutely irreducible. As a consequence, the hypersurface $\mathcal{H}_{\mathrm{nons}} = Z(P_{\mathrm{nons}})$ contains all the $\boldsymbol{f} \in \mathbb{P}_{\overline{K}}^{\boldsymbol{D}}$ for which $Z(\boldsymbol{f})$ is not absolutely irreducible. Below we describe a hypersurface in $\mathbb{P}_{\overline{K}}^{\boldsymbol{D}}$ of lower degree which contains all these systems (Theorem 4.5).

In Benoist (2012), Theorem 1.3, it is shown that the set of $\boldsymbol{f} \in \mathbb{P}_{\overline{K}}^{\boldsymbol{D}}$ for which the variety $Z(\boldsymbol{f}) \subset \mathbb{P}_{\overline{K}}^{n}$ is not a nonsingular complete intersection of dimension $n-s$ and degree δ is a hypersurface of $\mathbb{P}_{\overline{K}}^{\boldsymbol{D}}$. Furthermore, the author determines exactly the degrees of this hypersurface. As mentioned in the introduction to this paper, this result is achieved by combining a study of dual varieties of nonsingular toric varieties in characteristic zero and projective duality theory in positive characteristic. In particular, for $s=1$ the Benoist bound becomes the Boole bound $(n+1)(d_1-1)^n$. On the other hand, the bound of Theorem 4.2 is $\big((n+1)d_1-1\big)(d_1-1)^{n-1}$ in this case, which is fairly close to the Boole bound.

4.1.1. *Smooth complete intersections defined over* $\mathbb{F}_q$. Next we apply Theorem 4.2 in the case $K=\mathbb{F}_q$. By Theorem 4.2 and Proposition 3.6, and with $p_{\boldsymbol{D}}$ from (3.4), we obtain a lower bound on the number of those systems that define a smooth complete intersection. For fixed geometric data, it increases with growing q.

Corollary 4.3. *In the notation (2.8), assume that* $q \geq s(\sigma+n)\sigma^{n-s}\delta/3$. *Let* N_{nons} *be the number of* $\boldsymbol{f} \in \mathbb{P}^{\boldsymbol{D}}(\mathbb{F}_q)$ *for which* $Z(\boldsymbol{f}) \subset \mathbb{P}_{\mathbb{F}}^{n}$ *is a nonsingular complete intersection of dimension* $n-s$ *and degree* $\delta = d_1 \cdots d_s$. *Then*

$$1 - \frac{s(\sigma+n)\sigma^{n-s}\delta}{q} \leq \frac{N_{\text{nons}}}{p_{\boldsymbol{D}}} < 1. \tag{4.4}$$

4.2. **Absolutely irreducible complete intersections.** With notations as in the previous section, in this section we obtain an estimate on the number of polynomial systems defined over an arbitrary field K such that the corresponding projective variety is an absolutely irreducible complete intersection. As the approach is similar to that of Sections 3.2 and 4.1, we shall be brief.

Let $J_{s+1}(\boldsymbol{F},X)$ and $J_{s+2}(\boldsymbol{F},X)$ be the Jacobian determinants defined in (4.1). Consider the incidence variety

(4.5)

$$W_{\text{irr}} = \{(\boldsymbol{f},x) \in \mathbb{P}_{\overline{K}}^{\boldsymbol{D}} \times \mathbb{P}_{\overline{K}}^{n} : \boldsymbol{f}(x)=\boldsymbol{0}, J_{s+1}(\boldsymbol{f})(x)=0, J_{s+2}(\boldsymbol{f})(x)=0\}.$$

Arguing as in the proof of Lemma 4.1, we obtain the following.

Lemma 4.4. *The polynomials* $J_{s+1}(\boldsymbol{F},X), J_{s+2}(\boldsymbol{F},X), F_1,\ldots,F_s$ *form a regular sequence of* $K[\text{coeffs}(\boldsymbol{F}),X]$.

Our next result asserts that for a generic s–tuple $\boldsymbol{f}$ as above, the corresponding variety is an absolutely irreducible complete intersection. We also provide estimates on the degree of a hypersurface of $\mathbb{P}_{\overline{K}}^{\boldsymbol{D}}$ containing the elements $\boldsymbol{f}$ for which $Z(\boldsymbol{f})$ is not absolutely irreducible.

Theorem 4.5. *There exists a nonzero multihomogeneous polynomial* $P_{\mathrm{irr}} \in K[\mathrm{coeffs}(\boldsymbol{F})]$ *with*

$$\deg_{\mathrm{coeffs}(F_i)} P_{\mathrm{irr}} \leq \sigma\delta(\frac{\sigma}{d_i}+2) \leq 3\sigma^2\delta$$

for $1 \leq i \leq s$ *such that for any* $\boldsymbol{f} \in \mathbb{P}^{\boldsymbol{D}}_{\overline{K}}$ *with* $P_{\mathrm{irr}}(\boldsymbol{f}) \neq 0$*, the variety* $Z(\boldsymbol{f}) \subset \mathbb{P}^n_{\overline{K}}$ *is an absolutely irreducible complete intersection of dimension* $n-s$ *and degree* δ*.*

Proof. Lemma 4.4 shows that W_{irr} is of pure dimension $|\boldsymbol{D}|+n-s-2$. Its subvariety $W'_{\mathrm{irr}} = W_{\mathrm{irr}} \cap \{X_{s+2} = \cdots = X_n = 0\}$ may be seen as an incidence variety analogous to (4.2) associated to generic polynomials of $K[X_0,\ldots,X_{s+1}]$ of degrees $d_1,\ldots,d_s$. Therefore, applying Lemma 4.1 with $n = s+1$ we deduce that W'_{irr} is of pure dimension $|\boldsymbol{D}|-1$.

Furthermore, we let $\pi : \mathbb{P}^{\boldsymbol{D}}_{\overline{K}} \times \mathbb{P}^n_{\overline{K}} \to \mathbb{P}^{\boldsymbol{D}}_{\overline{K}}$ be the projection to the first argument. As π is a closed mapping, $\pi(W'_{\mathrm{irr}})$ is a closed subset of $\mathbb{P}^{\boldsymbol{D}}_{\overline{K}}$ of dimension at most $|\boldsymbol{D}|-1$ and hence there exists $\boldsymbol{f} \in \mathbb{P}^{\boldsymbol{D}}_{\overline{K}} \setminus \pi(W'_{\mathrm{irr}})$. For such an $\boldsymbol{f}$, the equations $\{\boldsymbol{f}=0, J_{s+1}(\boldsymbol{f})=0, J_{s+2}(\boldsymbol{f})=0, X_{s+2}=0,\ldots,X_n=0\}$ define the empty projective set.

This shows that the multivariate resultant $P \in K[\mathrm{coeffs}(\boldsymbol{F})]$ of formal polynomials $F_1,\ldots,F_s$ of degrees $d_1,\ldots,d_s$ and the polynomials $J_{s+1}(\boldsymbol{F},X), J_{s+2}(\boldsymbol{F},X), X_{s+2},\ldots,X_n$ is nonzero. Denote by $\mathcal{H}_{\mathrm{irr}} \subset \mathbb{P}^{\boldsymbol{D}}_{\overline{K}}$ the hypersurface defined by P and let $P_{\mathrm{irr}} \in K[\mathrm{coeffs}(\boldsymbol{F})]$ be any polynomial of minimal degree defining $\mathcal{H}_{\mathrm{irr}}$. We claim that the multihomogeneous polynomial P_{irr} satisfies the requirements of the theorem.

Indeed, let $\boldsymbol{f} = (f_1,\ldots,f_s) \in \mathbb{P}^{\boldsymbol{D}}_{\overline{K}}$ be such that $P_{\mathrm{irr}}(\boldsymbol{f}) \neq 0$, that is $\boldsymbol{f} \notin \mathcal{H}_{\mathrm{irr}}$. Observe that

$$\mathcal{H}_{\mathrm{irr}} = \pi(W'_{\mathrm{irr}}) = \pi\big(W_{\mathrm{irr}} \cap \{X_{s+2} = \cdots = X_n = 0\}\big).$$

For any $\boldsymbol{f} \in \mathbb{P}^{\boldsymbol{D}}_{\overline{K}}$, if $Z\big(\boldsymbol{f}, J_{s+1}(\boldsymbol{f}), J_{s+2}(\boldsymbol{f})\big) \subset \mathbb{P}^n_{\overline{K}}$ has dimension strictly greater than $n-s-2$, then $Z\big(\boldsymbol{f}, J_{s+1}(\boldsymbol{f}), J_{s+2}(\boldsymbol{f}), X_{s+2},\ldots,X_n\big)$ is nonempty, and the multivariate resultant of the polynomials $\boldsymbol{f}, J_{s+1}(\boldsymbol{f}), J_{s+2}(\boldsymbol{f}), X_{s+2},\ldots,X_n$ vanishes, that is, $\boldsymbol{f}$ belongs to $\mathcal{H}_{\mathrm{irr}}$. We conclude that, if $\boldsymbol{f} \notin \mathcal{H}_{\mathrm{irr}}$, then $Z\big(\boldsymbol{f}, J_{s+1}(\boldsymbol{f}), J_{s+2}(\boldsymbol{f})\big)$ is of pure dimension $n-s-2$. It follows that $Z(\boldsymbol{f})$ has dimension $n-s$ and $f_1,\ldots,f_s$ form a regular sequence of $\overline{K}[X]$. Furthermore, Eisenbud (1995), Theorem 18.15, proves that $f_1,\ldots,f_s$ generate a radical ideal of $\overline{K}[X]$. In particular, the singular locus of $Z(\boldsymbol{f})$ is contained in $\{\boldsymbol{f}=0, J_{s+1}(\boldsymbol{f})=0, J_{s+2}(\boldsymbol{f})=0\}$, which has dimension $s-2$, showing that $Z(\boldsymbol{f})$ is a normal complete intersection, and thus absolutely irreducible by Fact 2.1.

Now we estimate the multidegree of $\mathcal{H}_{\mathrm{irr}}$. For this purpose, we consider the class $[W'_{\mathrm{irr}}]$ of W'_{irr} in the Chow ring $\mathcal{A}^*(\mathbb{P}^{\boldsymbol{D}}_{\overline{K}} \times \mathbb{P}^n_{\overline{K}})$ of $\mathbb{P}^{\boldsymbol{D}}_{\overline{K}} \times \mathbb{P}^n_{\overline{K}}$. Denote by θ_i the class of the inverse image of a hyperplane of $\mathbb{P}^{D_i}_{\overline{K}}$ under the ith canonical projection $\mathbb{P}^{\boldsymbol{D}}_{\overline{K}} \times \mathbb{P}^n_{\overline{K}} \to \mathbb{P}^{D_i}_{\overline{K}}$ for $1 \leq i \leq s$ and by θ_0 the class of the inverse image of a hyperplane of $\mathbb{P}^n_{\overline{K}}$ under the projection $\mathbb{P}^{\boldsymbol{D}}_{\overline{K}} \times \mathbb{P}^n_{\overline{K}} \to \mathbb{P}^n_{\overline{K}}$ to the second argument. By the definition (4.5) of W_{irr} and the multihomogeneous Bézout theorem (2.4), we obtain

$$\begin{aligned}[W'_{\mathrm{irr}}] &= \Big(\prod_{i=1}^{s}(d_i\theta_0+\theta_i)\Big)\,(\sigma\theta_0+\theta_1+\cdots+\theta_s)^2\,\theta_0^{n-s-1}\\ &= \sigma^2\delta\,\theta_0^{n+1}+\sigma\delta\sum_{1\le i\le s}(\frac{\sigma}{d_i}+2)\theta_0^n\theta_i+\mathcal{O}(\theta_0^{n-1}),\end{aligned}$$

where $\mathcal{O}\big(\theta_0^{n-1}\big)$ is a sum of terms of degree at most $n-1$ in θ_0.

On the other hand, by definition $[\mathcal{H}_{\mathrm{irr}}] = [\pi(W'_{\mathrm{irr}})] = \deg_{\mathrm{coeffs}(F_1)} P_{\mathrm{irr}}\,\theta_1 + \cdots + \deg_{\mathrm{coeffs}(F_s)} P_{\mathrm{irr}}\,\theta_s$. Let $\jmath : \mathcal{A}^*\big(\mathbb{P}^{\boldsymbol{D}}_{\overline{K}}\big) \hookrightarrow \mathcal{A}^*\big(\mathbb{P}^{\boldsymbol{D}}_{\overline{K}} \times \mathbb{P}^n_{\overline{K}}\big)$ be the injective $\mathbb{Z}$–map $Q \mapsto \theta_0^n Q$ induced by π. Then (2.5) shows that $\jmath([\pi(W'_{\mathrm{irr}})]) \le [W'_{\mathrm{irr}}]$, where the inequality is understood in a coefficient–wise sense. This implies that, for $1 \le i \le s$, the following inequality holds:

$$\deg_{\mathrm{coeffs}(F_i)} P_{\mathrm{irr}} \le \sigma\delta\big(\frac{\sigma}{d_i}+2\big) \le 3\sigma^2\delta. \tag{4.6}$$

This completes the proof of the theorem. □

The hypersurface $\mathcal{H}_{\mathrm{irr}} \subset \mathbb{P}^{\boldsymbol{D}}$ of the proof of Theorem 4.5 is defined by the multivariate resultant $P = P^{[0,\ldots,s+1]} \in K[\mathrm{coeffs}(\boldsymbol{F})]$ of formal polynomials $F_1,\ldots,F_s$ of degrees $d_1,\ldots,d_s$ and the polynomials $J_{s+1}(\boldsymbol{F})$, $J_{s+2}(\boldsymbol{F})$, $X_{s+2},\ldots,X_n$. It is well-known that P is actually the multivariate resultant of the polynomials $F_i(X_0,\ldots,X_{s+1},0,\ldots,0)$ for $1 \le i \le s$ and $J_k(\boldsymbol{F})(X_0,\ldots,X_{s+1},0,\ldots,0)$ for $s < k \le s+2$; see, e.g., Cox *et al.* (1998), §3.3, Exercise 12. In particular, P only depends on the coefficients $F_{i,\boldsymbol{\alpha}}$ with $\alpha_k = 0$ for $s+2 \le k \le n$. By considering the sets of indices $[0,\ldots,s,k]$ for $s < k \le n$, one obtains multivariate resultants $P^{[0,\ldots,s,k]} \in K[\mathrm{coeffs}(\boldsymbol{F})]$ whose set of common zeros in $\mathbb{P}^{\boldsymbol{D}}_{\overline{K}}$ contains all the $\boldsymbol{f}$ not defining a normal complete intersection of dimension $n-s$ and degree δ. Furthermore, it can be proved that the polynomials $P^{[0,\ldots,s,k]}$ for $s < k \le n$ form a regular sequence of $K[\mathrm{coeffs}(\boldsymbol{F})]$. This shows that the set of $\boldsymbol{f} \in \mathbb{P}^{\boldsymbol{D}}_{\overline{K}}$ that do not define a normal complete intersection of dimension $n-s$ and degree δ is contained in a subvariety of $\mathbb{P}^{\boldsymbol{D}}_{\overline{K}}$ of pure codimension $n-s$.

4.2.1. *Absolutely irreducible complete intersections defined over* $\mathbb{F}_q$. Now we apply Theorem 4.5 in the case $K = \mathbb{F}_q$. Combining Theorem 4.5 and Proposition 3.6, we can bound the number of polynomial systems as above defining absolutely irreducible complete intersections.

Corollary 4.6. *Suppose that* $q \ge s\sigma^2\delta$. *Let* $N^{\boldsymbol{d}}_{\mathrm{irr}}$ *be the number of* $\boldsymbol{f} \in \mathbb{P}^{\boldsymbol{D}}(\mathbb{F}_q)$ *such that* $Z(\boldsymbol{f}) \subset \mathbb{P}^n_{\mathbb{F}}$ *is an absolutely irreducible complete* *intersection of dimension* $n-s$ *and degree* $\delta = d_1 \cdots d_s$. *Then*

$$1 - \frac{3s\sigma^2\delta}{q} \le \frac{N_{\mathrm{irr}}}{p_{\boldsymbol{D}}} < 1. \tag{4.7}$$

5. Most systems define a degenerate variety

We fix the dimension $n \ge 2$ of a projective ambient space $\mathbb{P}^n_{\overline{K}}$ over an algebraic closure $\overline{K}$ of a field K, the codimension s with $1 \le s < n$ and the degree $b > 0$, and discuss geometric properties which are satisfied by "most" complete intersections with these features. We show that most complete intersections in this sense are absolutely irreducible hypersurfaces within some linear projective subspace; in particular, they are degenerate for $s \ge 2$. In fact, this means that most systems define a "maximally degenerate" variety V; unless V is linear, its points span a linear space of dimension at least $\dim V + 1$. We also provide estimates on the number of polynomial systems defined over a finite field $\mathbb{F}_q$ which fail to define such an absolutely irreducible hypersurface.

In the previous sections, we fix a degree pattern $\boldsymbol{d} = (d_1, \ldots, d_s)$ and consider the corresponding variety $\mathbb{P}^{\boldsymbol{D}}_{\overline{K}}$. Then "most" polynomial sequences in this variety turn out to have the desired properties. This is in the usual sense of algebraic geometry of comprising all polynomials outside of a fixed proper closed subvariety.

The spirit of this section is different, more combinatorial than geometric. For fixed s and b, we consider all polynomial sequences of any degree pattern $\boldsymbol{d}$ for which $d_1 \cdots d_s = b$. These sequences do not seem to form an algebraic set in a natural way, but rather are a disjoint union of several varieties. We determine which of them has the largest dimension and call its degree pattern "typical". This does not correspond to the geometric notion of "most" in the sense of the other sections.

But it is a useful tool to understand the situation over a finite field. Given s and b, we have a finite set of polynomial sequences and find the type of variety that their majority determines. This turns out to be (irreducible smooth) hypersurfaces in some linear projective subspace. We may thus say that most sequences determine degenerate varieties (for $s \ge 2$).

More precisely, we consider the multiprojective variety $S^{(b)}$ of all systems $\boldsymbol{f} = (f_1, \ldots, f_s)$ of homogeneous polynomials with $1 \le \deg f_i \le b$ for all i. Given a degree pattern $\boldsymbol{d} = (d_1, \ldots, d_s) \in \mathbb{N}^s$ with $d_1 \ge d_2 \ge \cdots \ge d_s \ge 1$ and $d_1 \cdots d_s = b$, the systems $\boldsymbol{f}$ with degree pattern $\boldsymbol{d}$ form a closed subvariety $S_{\boldsymbol{d}}$ of $S^{(b)}$. Their union $S = \bigcup_{\boldsymbol{d}} S_{\boldsymbol{d}}$ over all such $\boldsymbol{d}$ is the object studied in this section. We show that for $\boldsymbol{d}^{(b)} = (b, 1, \ldots, 1)$, $S_{\boldsymbol{d}^{(b)}}$ is the unique component of S with maximal dimension. All systems in $S_{\boldsymbol{d}^{(b)}}$ describe a hypersurface within a linear subspace of codimension $s-1$, which is proper if $s \ge 2$.

A result of a similar flavor was shown by Eisenbud & Harris (1992). They prove that in the Chow variety of curves of degree b in $\mathbb{P}^n_{\overline{K}}$, most curves are planar and irreducible if $4n-8 \leq b$. Based on this approach, Cesaratto *et al.* (2013) provide numerical bounds for the probability that a curve randomly chosen in the Chow variety over a finite field is planar and irreducible. At first sight, it may look surprising that a generic curve in this sense is planar. We show a corresponding result for more general varieties: the dimension of the variety of polynomial systems defining absolutely irreducible hypersurfaces within some linear projective subspace is larger than the dimension of systems defining other types of varieties.

The two models of varieties are different: we consider defining systems of polynomials, while Eisenbud & Harris (1992) and Cesaratto *et al.* (2013) deal with varieties themselves. In their case of curves, they find that unions of lines form a component of maximal dimension within the Chow variety if $b < 4n - 8$. The corresponding unions of linear subspaces do not turn up in our approach.

5.1. **Dimension of systems with a given Bézout number.** Assume that $s \geq 2$ and for any $\boldsymbol{d} = (d_1, \ldots, d_s)$ with $d_1 \geq d_2 \geq \cdots \geq d_s \geq 1$ and $d_1 \geq 2$, let $S_{\boldsymbol{d}}$ be the multiprojective variety of all homogeneous $f_1, \ldots, f_s \in K[X_0, \ldots, X_n]$ with $\deg f_i = d_i$ for all i. The Bézout number of such a system is $\delta(\boldsymbol{d}) = d_1 \cdots d_s$. According to Theorems 3.5, 4.2, and 4.5, the projective variety $V = Z(\boldsymbol{f}) \subset \mathbb{P}^n_{\overline{K}}$ defined by a generic $\boldsymbol{f} = (f_1, \ldots, f_s)$ is a smooth absolutely irreducible complete intersection of dimension $n - s$ and degree $\delta(\boldsymbol{d})$. As the degree pattern $(d_1, \ldots, d_s)$ is not fixed a priori, one may wonder how frequently a given pattern arises. We shall show that the most typical pattern is that corresponding to hypersurfaces, namely, $(b, 1, \ldots, 1)$.

For this purpose, for any degree pattern $\boldsymbol{d} = (d_1, \ldots, d_s) \in \mathbb{N}^s$ as above, we abbreviate

$$\delta(\boldsymbol{d}) = d_1 \cdots d_s, \quad D_i(\boldsymbol{d}) = \binom{d_i + n}{n} - 1 \quad \text{for } 1 \leq i \leq s,$$
$$\boldsymbol{D}(\boldsymbol{d}) = (D_1(\boldsymbol{d}), \ldots, D_s(\boldsymbol{d})), \quad |\boldsymbol{D}(\boldsymbol{d})| = D_1(\boldsymbol{d}) + \cdots + D_s(\boldsymbol{d}).$$

This notation is in agreement with that of (2.8), where the dependence on $\boldsymbol{d}$ is not explicitly indicated, since we were considering a fixed degree pattern.

We consider the hypersurface degree pattern $\boldsymbol{d}^{(b)} = (b, 1, \ldots, 1) \in \mathbb{N}^s$ and $\boldsymbol{D}^{(b)} = \boldsymbol{D}(\boldsymbol{d}^{(b)})$ and start with the following result.

Lemma 5.1. *We have* $|\boldsymbol{D}^{(b)}| > |\boldsymbol{D}(\boldsymbol{d})|$ *for all* $\boldsymbol{d} \neq \boldsymbol{d}^{(b)}$ *with* $\delta(\boldsymbol{d}) = b$.

Proof. An elementary calculation shows that for $a \geq 2$ we have

$$\frac{(2a+2)!}{(a+2)!} > 2\frac{(2a)!}{a!}.$$

It follows that

$$\frac{(2a+n)!}{(a+n)!} > 2\frac{(2a)!}{a!}$$

for $n \geq 2$, since the left–hand side is monotonically increasing in n. Next, we have for $a \geq c \geq 2$ that

$$\binom{a+n}{n} \geq \binom{c+n}{n},$$

$$\frac{(ac+n)!}{(ac)!} > 2\frac{(a+n)!}{a!}. \tag{5.1}$$

Dividing both sides by $n!$, we find that with $s = 2$,

$$|\boldsymbol{D}(bc, 1)| > |\boldsymbol{D}(b, c)|. \tag{5.2}$$

The general claim of the lemma follows by induction on s. □

Let $a \geq c \geq 2$ and let ρ be a prime number dividing c. From (5.1) one deduces that $|\boldsymbol{D}(a\rho, c/\rho)| > |\boldsymbol{D}(a, c)|$. For an integer b, we set $g(b) = 0$ if b is prime, and otherwise

(5.3)

$$g(b) = |\boldsymbol{D}^{(b)}| - |\boldsymbol{D}(b/\rho, \rho, 1, \ldots, 1)| = \binom{b+n}{n} - \binom{b/\rho+n}{n} - \binom{\rho+n}{n},$$

where ρ is the smallest prime number dividing b. Extending the binomial $u(\tau) = \binom{b/\tau+n}{n}$ to a real function of the real variable τ on the interval $[2\,..\,b/2]$ via the gamma function, u is convex and assumes its maximum at one of the two endpoints of the interval, namely, at $\tau = 2$; see von zur Gathen (2011), (3.6). It follows that

$$g(b) \geq \binom{b+n}{n} - 2\binom{b/2+n}{n}. \tag{5.4}$$

We always have $g(b) \geq 1$, but $g(b)$ may be quite large. For example, if $b^2 \geq 2n^3$, then $g(b) \geq b^2/2n^2$. Furthermore, for $n > s > 1$ and $b \geq 2$ composite, we have

$$|\boldsymbol{D}(b/\rho, \rho, 1, \ldots, 1)| = \max_{\delta(\boldsymbol{d})=b,\, \boldsymbol{d}\neq\boldsymbol{d}^{(b)}} |\boldsymbol{D}(\boldsymbol{d})|,$$

and

$$|\boldsymbol{D}^{(b)}| \geq |\boldsymbol{D}(\boldsymbol{d})| + g(b) \tag{5.5}$$

for any $\boldsymbol{d}$ with $\delta(\boldsymbol{d}) = b$ and $\boldsymbol{d} \neq \boldsymbol{d}^{(b)}$.

Combining Lemma 5.1 and (5.5), we can conclude that among all s–tuples of homogeneous polynomials having a degree pattern $\boldsymbol{d}$ with $\delta(\boldsymbol{d}) = b$, "most" of them define a hypersurface within some linear projective subspace of $\mathbb{P}^n_{\overline{K}}$. More precisely, we have the following result.

Corollary 5.2. *Let $n \geq 2$, $1 \leq s < n$, and $b > 0$. For any degree pattern $\boldsymbol{d} \neq \boldsymbol{d}^{(b)}$ with $\delta(\boldsymbol{d}) = b$,*

$$\dim \mathbb{P}_{\overline{K}}^{\boldsymbol{D}^{(b)}} \geq \dim \mathbb{P}_{\overline{K}}^{\boldsymbol{D}(\boldsymbol{d})} + g(b).$$

Proof. Since $\dim \mathbb{P}_{\overline{K}}^{\boldsymbol{D}(\boldsymbol{d})} = |\boldsymbol{D}(\boldsymbol{d})|$ for any degree pattern $\boldsymbol{d}$, the corollary follows from (5.5). □

We may strengthen the conclusions of Corollary 5.2 by applying Theorem 4.5.

Corollary 5.3. *With assumptions as in Corollary 5.2, denote by* $\mathcal{S}_{\mathrm{irr}}^{\boldsymbol{d}}$ *the set of* $\boldsymbol{f} \in \mathbb{P}_{\overline{K}}^{\boldsymbol{D}(\boldsymbol{d})}$ *with degree pattern* $\boldsymbol{d}$ *such that* $Z(\boldsymbol{f}) \subset \mathbb{P}_{\overline{K}}^{n}$ *is an absolutely irreducible complete intersection of dimension* $n-s$ *and degree* b*. Then for any degree pattern* $\boldsymbol{d} \neq \boldsymbol{d}^{(b)}$ *with* $\delta(\boldsymbol{d}) = b$*, we have*

$$\dim \mathcal{S}_{\mathrm{irr}}^{\boldsymbol{d}^{(b)}} \geq \dim \mathcal{S}_{\mathrm{irr}}^{\boldsymbol{d}} + g(b).$$

Proof. Let $\boldsymbol{d}$ be a degree pattern with $\delta(\boldsymbol{d}) = b$. According to Theorem 4.5, there exists a hypersurface $\mathcal{H}_{\mathrm{irr}}^{\boldsymbol{d}} \subset \mathbb{P}_{\overline{K}}^{\boldsymbol{D}(\boldsymbol{d})}$ such that $Z(\boldsymbol{f}) \subset \mathbb{P}_{\overline{K}}^{n}$ is an absolutely irreducible complete intersection of dimension $n-s$ and degree b for any $\boldsymbol{f} \in \mathbb{P}_{\overline{K}}^{\boldsymbol{D}(\boldsymbol{d})} \setminus \mathcal{H}_{\mathrm{irr}}$. This implies that

$$\dim \mathcal{S}_{\mathrm{irr}}^{\boldsymbol{d}} = \dim \mathbb{P}_{\overline{K}}^{\boldsymbol{D}(\boldsymbol{d})} = |\boldsymbol{D}(\boldsymbol{d})|.$$

The conclusion now follows from Corollary 5.2. □

5.2. **Systems defined over a finite field.** In this section we obtain a quantitative version of Corollary 5.3 for the set of s–tuples of homogeneous polynomials with coefficients in $\mathbb{F}_q$ having any degree pattern $\boldsymbol{d}$ with $\delta(\boldsymbol{d}) = b$. For the case $s = 1$, von zur Gathen *et al.* (2013), Corollary 6.8, shows that the number N_{irr}^{1} of homogeneous polynomials $f_1 \in \mathbb{F}_q[X_0, \ldots, X_n]$ of degree b which are absolutely irreducible satisfies the following estimate:

$$\left| N_{\mathrm{irr}}^{1} - \frac{q^{\binom{b+n}{n}} - q^{\binom{b+n-1}{n}}}{q-1} \right| \leq 4\, q^{\binom{b+n-1}{n}+n-1} \frac{1-q^{-n}}{(1-q^{-1})^2},$$

where the 4 can be replaced by 3 for $n \geq 3$.

We denote as $M_s(b)$ the number of $\boldsymbol{d}$ as in (2.8) with $\delta(\boldsymbol{d}) = b$, which equals the number of nontrivial unordered factorizations of b with at most s factors, and first estimate this quantity.

Lemma 5.4. *For positive integers* $b \geq 2$ *and* s*, we have* $M_s(b) \leq b^{\log_2 \log_2 b}$*.*

Proof. We consider *unordered factorizations* F of $b \in \mathbb{N}$ with s factors. Such an F is a multiset of s positive integers whose product (with multiplicities) equals b. The number 1 is allowed as a factor. Formally, we have $F \colon \mathbb{N}_{\geq 1} \to \mathbb{N}$ with $\prod_{a \in \mathbb{N}_{\geq 1}} a^{F(a)} = b$ and $\sum_{a \in \mathbb{N}_{\geq 1}} F(a) = s$. Then a "occurs $F(a)$ times" in F, and a "occurs" in F if $F(a) \geq 1$.

Picking primes p and q with $p \mid b$ and $q \nmid b$, we take for any F some a occurring in F with $p \mid a$ and replace one copy of a by aq/p. This new multiset F' is a factorization of bq/p. Replacing the unique occurrence of a multiple of q in any factorization of bq/p by the same multiple of

p gives a factorization of b. When applied to F', it yields the original F. Thus the map $F \mapsto F'$ is injective and $M_s(b) \leq M_s(bq/p)$.

Let $m = \Omega(b)$ be the number of prime factors of b, counted with multiplicities, and c any squarefree integer with $m = \Omega(c)$ prime factors. The above shows that $M_s(b) \leq M_s(c)$. A factorization of c corresponds to a partition of $\{1, \ldots, m\}$ into $t \leq s$ disjoint nonempty subsets, together with $s-t$ times the empty set (meaning $F(1) = s-t$ in the above notation). We drop the restriction $t \leq s$ and consider all partitions of $\{1, \ldots, m\}$ into nonempty subsets. The number of such partitions is the mth Bell number B_m. Since $M_s(2) = 1$, we may assume that $m > 2$. By Berend & Tassa (2010), we have

$$\log_2 B_m \leq m \cdot \log_2(0.8m/\ln m) < m \log_2 m.$$

Since $m = \Omega(b)$, we have $2^m \leq b$. It follows that

$$M_s(b) \leq M_s(c) \leq B_m < 2^{m \log_2 m} \leq b^{\log_2 \log_2 b}.$$

□

Combining Lemma 5.1 and (5.5) we obtain an estimate on the number of polynomials systems as above defining a complete intersection which is a hypersurface in some linear subspace. As a special case of (3.4), the number of s–tuples $\boldsymbol{f} = (f_1, \ldots, f_s)$ of homogeneous polynomials of $\mathbb{F}_q[X_0, \ldots, X_n]$ with degree pattern $(b, 1, \ldots, 1)$, up to nonzero multiples in $\mathbb{F}_q$ of any f_i, is equal to

$$\#\mathbb{P}^{\boldsymbol{D}^{(b)}}(\mathbb{F}_q) = \#\mathbb{P}^{D_b}(\mathbb{F}_q) \cdot \big(\#\mathbb{P}^n(\mathbb{F}_q)\big)^{s-1} = p_{D_b} p_n^{s-1},$$

where $D_b = \binom{b+n}{n} - 1$. The estimates in the following will be expressed as a deviation from this value. For any $\boldsymbol{d} \neq \boldsymbol{d}^{(b)}$ with $\delta(\boldsymbol{d}) = b$, (5.5) implies that

$$p_{\boldsymbol{D}(\boldsymbol{d})} \leq \frac{p_{D_b} p_n^{s-1}}{q^{g(b)}}. \tag{5.6}$$

Theorem 5.5. *Let N_{hyp} denote the number of $\boldsymbol{f} \in \mathbb{P}^{\boldsymbol{D}(\boldsymbol{d})}(\mathbb{F}_q)$ defining a complete intersection $Z(\boldsymbol{f}) \subset \mathbb{P}^n_{\mathbb{F}}$ of dimension $n-s$ and degree b, which is a hypersurface in some linear projective subspace of $\mathbb{P}^n_{\mathbb{F}}$ for some $\boldsymbol{d}$ as in (2.8) with $\delta(\boldsymbol{d}) = b$. Then*

$$\Big|\frac{N_{\text{hyp}}}{p_{D_b} p_n^{s-1}} - 1\Big| \leq \frac{1 + 9q^{-1}}{q^{n-s+3}} + \frac{M_s(b)}{q^{g(b)}} \leq \frac{1 + 9q^{-1}}{q^{n-s+3}} + \frac{b^{\log_2 \log_2 b}}{q^{g(b)}}.$$

Proof. Let $\boldsymbol{d} = (d_1, \ldots, d_s) \in \mathbb{N}^s$ be a degree pattern with $d_1 \geq d_2 \geq \cdots \geq d_s \geq 1$, $\delta(\boldsymbol{d}) = b$ and $\boldsymbol{d} \neq \boldsymbol{d}^{(b)}$. Denote by $N_{\text{ci}}^{\boldsymbol{d}}$ the number of $\boldsymbol{f} \in \mathbb{P}^{\boldsymbol{D}(\boldsymbol{d})}(\mathbb{F}_q)$ defining a complete intersection of $\mathbb{P}^n_{\mathbb{F}}$ of dimension $n-s$ and degree $b = \delta(\boldsymbol{b})$. We have the obvious upper bound

$$N_{\text{ci}}^{\boldsymbol{d}} \leq p_{\boldsymbol{D}(\boldsymbol{d})} = \prod_{1 \leq i \leq s} p_{D_i(\boldsymbol{d})}.$$

Any complete intersection with degree pattern $\boldsymbol{d}^{(b)}$ is a hypersurface in some linear projective subspace. Therefore,

$$\left|N_{\mathrm{hyp}} - p_{D_b} p_n^{s-1}\right| \leq \left|N_{\mathrm{ci}}^{\boldsymbol{d}^{(b)}} - p_{D_b} p_n^{s-1}\right| + \sum_{\substack{\delta(\boldsymbol{d})=b \\ \boldsymbol{d}\neq \boldsymbol{d}^{(b)}}} p_{\boldsymbol{D}(\boldsymbol{d})}.$$

The sum leads to the second summands in the bounds of the theorem via (5.6). We now consider the first term on the right–hand side of the inequality. Let $\boldsymbol{f} = (f_1, \ldots, f_s)$ be an s–tuple of polynomials of $\mathbb{F}_q[X_0, \ldots, X_n]$ with degree pattern $\boldsymbol{d}^{(b)}$. We first consider the case where $f_2 = X_0, \ldots, f_s = X_{s-2}$. Then $Z(\boldsymbol{f}) \subset \mathbb{P}^n_{\mathbb{F}}$ is a complete intersection of dimension $n-s$ and degree b if and only if $g_1 = f_1(0, \ldots, 0, X_{s-1}, \ldots, X_n)$ is a squarefree polynomial of $\mathbb{F}_q[X_{s-1}, \ldots, X_n]$.

We fix a monomial order of $\mathbb{F}_q[X_{s-1}, \ldots, X_n]$ and normalize g_1 by requiring that its leading coefficient with respect to this order be equal to 1. Denote by $S_{n-s+1,b}(\mathbb{F}_q)$ the set of normalized squarefree homogeneous polynomials of $\mathbb{F}_q[X_{s-1}, \ldots, X_n]$ of degree b. Then

$$\#S_{n-s+1,b}(\mathbb{F}_q) \cdot q^{\binom{b+n}{n} - \binom{b+n-s+1}{n-s+1}} = \#\{Z(\boldsymbol{f}) \subset \mathbb{P}^n_{\mathbb{F}} \text{ complete intersections of degree } b\colon f_2 = X_0, \ldots, f_s = X_{s-2}\}.$$

We can make the previous argument for any sequence $(f_2, \ldots, f_s) \in (\mathbb{P}^n_{\mathbb{F}}(\mathbb{F}_q))^{s-1}$ with $f_2, \ldots, f_s$ linearly independent linear forms, that is, for any $(s-1)$–tuple $(f_2, \ldots, f_s)$ of homogeneous polynomials of $\mathbb{F}_q[X_0, \ldots, X_n]$ with degree pattern $(1, \ldots, 1)$ such that $f_2, \ldots, f_s$ are linearly independent, up to multiples in $\mathbb{F}_q$ of any f_i. If N_{ind} is the number of elements $(f_2, \ldots, f_s) \in (\mathbb{P}^n_{\mathbb{F}}(\mathbb{F}_q))^{s-1}$ with $f_2, \ldots, f_s$ linearly independent, then

$$N_{\mathrm{ci}}^{\boldsymbol{d}^{(b)}} = \#S_{n-s+1,b}(\mathbb{F}_q) \cdot q^{\binom{b+n}{n} - \binom{b+n-s+1}{n-s+1}} \cdot N_{\mathrm{ind}}, \tag{5.7}$$

$$N_{\mathrm{ind}} = \prod_{0\leq k\leq s-2} \frac{q^{n+1} - q^k}{q-1} = p_n^{s-1} \prod_{1\leq k\leq s-2} \frac{q^{n+1} - q^k}{q^{n+1} - 1} \leq p_n^{s-1}. \tag{5.8}$$

According to von zur Gathen *et al.* (2013), Corollary 5.7, we have

$$\left|\#S_{n-s+1,b}(\mathbb{F}_q) - \frac{q^{\binom{b+n-s+1}{n-s+1}}}{q-1}\right| \leq \frac{3\, q^{\binom{b+n-s-1}{n-s+1} + n - s}}{(1-q^{-1})^2}.$$

Therefore, we find that

$$\begin{aligned}
\left|N_{\mathrm{ci}}^{\boldsymbol{d}^{(b)}} - \frac{q^{\binom{b+n}{n}}}{q-1} N_{\mathrm{ind}}\right| &\leq \frac{3\, q^{\binom{b+n}{n} - \binom{b+n-s+1}{n-s+1} + \binom{b+n-s-1}{n-s+1} + n-s}}{(1-q^{-1})^2} N_{\mathrm{ind}} \\
&\leq \frac{3\, q^{\binom{b+n}{n} - \frac{b^2+n-s}{b+n-s}\binom{b+n-s}{n-s} + n-s}}{(1-q^{-1})^2} p_n^{s-1}.
\end{aligned}$$

Observe that, for $b \geq 2$ and $n-s \geq 2$,

$$\frac{b^2+n-s}{b+n-s} \binom{b+n-s}{n-s} - n + s \geq n - s + 5. \tag{5.9}$$

Indeed, as the left–hand side is monotonically increasing in b, it suffices to consider $b = 2$. For $b = 2$ and $n - s \geq 2$, an elementary calculation shows that (5.9) is satisfied. As a consequence, we obtain

$$\left|N_{\mathrm{ci}}^{\boldsymbol{d}^{(b)}} - \frac{q^{\binom{b+n}{n}}}{q-1}N_{\mathrm{ind}}\right| \leq \frac{3\,q^{\binom{b+n}{n}-n+s-5}}{(1-q^{-1})^2}p_n^{s-1} \leq \frac{13}{2}\,\frac{p_{D_b}p_n^{s-1}}{q^{n-s+4}}.$$

In order to get rid of the term N_{ind}, we use

$$\frac{q^{n+1}-q^k}{q^{n+1}-1} \geq 1 - \frac{1}{q^{n+1-k}}$$

for $1 \leq k \leq n+1$, and thus

$$1 \geq \prod_{1\leq k\leq s-2}\frac{q^{n+1}-q^k}{q^{n+1}-1} \geq \prod_{1\leq k\leq s-2}\big(1-\frac{1}{q^{n+1-k}}\big) \geq 1 - \frac{1+2q^{-1}}{q^{n-s+3}},$$

It follows that

$$p_n^{s-1} - q^{-n+s-3}(1+2q^{-1})p_n^{s-1} \leq N_{\mathrm{ind}} \leq p_n^{s-1}. \tag{5.10}$$

We deduce that

$$\begin{aligned}\left|N_{\mathrm{ci}}^{\boldsymbol{d}^{(b)}} - p_{D_b}p_n^{s-1}\right| &\leq \left|\frac{q^{\binom{b+n}{n}}}{q-1}N_{\mathrm{ind}} - p_{D_b}p_n^{s-1}\right| + \frac{13}{2}\,\frac{p_{D_b}p_n^{s-1}}{q^{n-s+4}} \\ &\leq \frac{1+2q^{-1}}{q^{n-s+3}}p_{D_b}p_n^{s-1} + \frac{p_n^{s-1}}{q-1} + \frac{13}{2}\,\frac{p_{D_b}p_n^{s-1}}{q^{n-s+4}}.\end{aligned}$$

The statement of the theorem readily follows. □

The error term in Theorem 5.5 decreases with growing q for fixed geometric data.

Next we estimate the number of polynomial systems as above defining an absolutely irreducible complete intersection. In view of Lemma 5.1 and Theorem 5.5, we have to pay particular attention to the degree pattern $\boldsymbol{d}^{(b)}$, which is the subject of the next result.

Lemma 5.6. *Let $N_{\mathrm{irr}}^{\boldsymbol{d}^{(b)}}$ be the number of $\boldsymbol{f} \in \mathbb{P}^{\boldsymbol{D}^{(b)}}(\mathbb{F}_q)$ defining an absolutely irreducible complete intersection $Z(\boldsymbol{f}) \subset \mathbb{P}_{\mathbb{F}}^n$ of dimension $n-s$ and degree b which is a hypersurface in some linear projective subspace of $\mathbb{P}_{\mathbb{F}}^n$. Then*

$$\left|\frac{N_{\mathrm{irr}}^{\boldsymbol{d}^{(b)}}}{p_{D_b}p_n^{s-1}} - 1\right| \leq \frac{1+14\,q^{-1}}{q^{n-s+3}}.$$

for $b > 2$ or $n-s > 3$. For $b = 2$ and $n - s \leq 3$, the statement holds with $1 + 14\,q^{-1}$ replaced by $14\,q^2$.

Proof. Let $\boldsymbol{f} = (f_1, \ldots, f_s)$ be an s–tuple of homogeneous polynomial of $\mathbb{F}_q[X_0, \ldots, X_n]$ with degree pattern $\boldsymbol{d}^{(b)}$. We first consider the case $f_2 = X_0, \ldots, f_s = X_{s-2}$. We have that $Z(\boldsymbol{f}) \subset \mathbb{P}_{\mathbb{F}}^n$ is absolutely irreducible if and only if $g_1 = f_1(0, \ldots, 0, X_{s-1}, \ldots, X_n)$ is an absolutely irreducible polynomial of $\mathbb{F}_q[X_{s-1},\ldots,X_n]$. We normalize g_1 by requiring that its leading coefficient with respect to a given monomial order of $\mathbb{F}_q[X_{s-1},\ldots,X_n]$ is equal to 1. Denote by $A_{n-s+1,b}(\mathbb{F}_q)$ the set of normalized absolutely irreducible polynomials of $\mathbb{F}_q[X_{s-1},\ldots,X_n]$ of degree b. We have

$$\#A_{n-s+1,b}(\mathbb{F}_q)\, q^{\binom{b+n}{n}-\binom{b+n-s+1}{n-s+1}} = \#\{Z(\boldsymbol{f})\subset \mathbb{P}^n_{\mathbb{F}} \text{ absolutely irreducible}: f_2=X_0,\ldots,f_s=X_{s-2}\}.$$

Now we let $(f_2,\ldots,f_s)$ run through all the sequences of linearly–independent linear forms of $\mathbb{F}_q[X_0,\ldots,X_n]$, namely through all the $(s-1)$–tuples of homogeneous polynomials of $\mathbb{F}_q[X_0,\ldots,X_n]$ of degree pattern $(1,\ldots,1)$ with $f_2,\ldots,f_s$ linearly independent, up to multiples in $\mathbb{F}_q$ of any f_i. If $N_{\rm ind}$ denotes the number of elements $(f_2,\ldots,f_s)\in(\mathbb{P}^n_{\mathbb{F}}(\mathbb{F}_q))^{s-1}$ with $f_2,\ldots,f_s$ linearly independent, then

$$N_{\rm irr}^{\boldsymbol{d}^{(b)}} = \#A_{n-s+1,b}(\mathbb{F}_q)\, q^{\binom{b+n}{n}-\binom{b+n-s+1}{n-s+1}} N_{\rm ind}.$$

From von zur Gathen *et al.* (2013), Corollary 6.8, we have that

$$\Big|\#A_{n-s+1,b}(\mathbb{F}_q) - \frac{q^{\binom{b+n-s+1}{n-s+1}}}{q-1}\Big| \le \frac{3\,q^{\binom{b+n-s}{n-s+1}+n-s}}{(1-q^{-1})^2}.$$

Further, by (5.10) we have $|N_{\rm ind}-p_n^{s-1}|\le q^{-n+s-3}(1+2q^{-1})p_n^{s-1}$. As a consequence,

$$\begin{aligned}\Big|N_{\rm irr}^{\boldsymbol{d}^{(b)}} - p_{D_b}p_n^{s-1}\Big| &\le \Big|N_{\rm irr}^{\boldsymbol{d}^{(b)}} - \#A_{n-s+1,b}(\mathbb{F}_q)\, q^{\binom{b+n}{n}-\binom{b+n-s+1}{n-s+1}}p_n^{s-1}\Big| \\ &\quad + \Big|\#A_{n-s+1,b}(\mathbb{F}_q)\, q^{\binom{b+n}{n}-\binom{b+n-s+1}{n-s+1}}p_n^{s-1} - p_{D_b}p_n^{s-1}\Big| \\ &\le p_{D_b}\left|N_{\rm ind}-p_n^{s-1}\right| \\ &\quad + p_{D_b}p_n^{s-1}\frac{q-1}{q^{\binom{b+n-s+1}{n-s+1}}}\Big|\#A_{n-s+1,b}(\mathbb{F}_q) - \frac{q^{\binom{b+n-s+1}{n-s+1}}}{q-1}\Big| \\ &\le \Big(\frac{1+2q^{-1}}{q^{n-s+3}} + 12\,q^{-\binom{b+n-s}{n-s}+n-s+1}\Big)\, p_{D_b}p_n^{s-1}.\end{aligned}$$

Finally, taking into account that

$$-\binom{b+n-s}{n-s} + n-s+1 \le -n+s-4$$

for $b>2$ or $n-s>3$, the statement of the lemma readily follows. □

Now we are ready to estimate the number of polynomials systems as above defining an absolutely irreducible projective subvariety of $\mathbb{P}^n_{\mathbb{F}}$ of dimension $n-s$ and degree b defined over $\mathbb{F}_q$. We recall $g(b)$ from (5.3).

Theorem 5.7. *Let $N^b_{\rm irr}$ be the number of $\boldsymbol{f}\in\mathbb{P}^{\boldsymbol{D}(\boldsymbol{d})}(\mathbb{F}_q)$ defining an absolutely irreducible complete intersection $Z(\boldsymbol{f})\subset\mathbb{P}^n_{\mathbb{F}}$ of dimension*

$n-s$ and degree b which is a hypersurface in some linear projective subspace of $\mathbb{P}_{\mathbb{F}}^n$, for any $\boldsymbol{d}$ with $\delta(\boldsymbol{d})=b$. Then

$$\left|\frac{N_{\mathrm{irr}}^b}{p_{D_b}p_n^{s-1}}-1\right|\le \frac{1+14\,q^{-1}}{q^{n-s+3}}+\frac{b^{\log_2\log_2 b}}{q^{g(b)}}.$$

if $b>2$ or $n-s>3$. For $b=2$ and $n-s\le 3$, the statement holds with $1+14\,q^{-1}$ replaced by $14\,q^2$.

Proof. Let $N_{\mathrm{irr}}^{\ne \boldsymbol{d}^{(b)}}$ denote the number of $\boldsymbol{f}\in\mathbb{P}^{\boldsymbol{D}(\boldsymbol{d})}(\mathbb{F}_q)$ such that $Z(\boldsymbol{f})$ is an absolutely irreducible complete intersection of dimension $n-s$ and degree b, not having degree pattern $\boldsymbol{d}^{(b)}$. We have

$$\left|N_{\mathrm{irr}}^b-p_{D_b}p_n^{s-1}\right|\le\left|N_{\mathrm{irr}}^{\boldsymbol{d}^{(b)}}-p_{D_b}p_n^{s-1}\right|+N_{\mathrm{irr}}^{\ne\boldsymbol{d}^{(b)}}.$$

On the one hand, Lemma 5.6 provides an upper bound for the first term in the right–hand side. On the other hand, by Lemmas 5.1 and 5.4 and (5.5), we find

$$\begin{aligned}N_{\mathrm{irr}}^{\ne\boldsymbol{d}^{(b)}}&\le\sum_{\substack{\delta(\boldsymbol{d})=b\\ \boldsymbol{d}\ne\boldsymbol{d}^{(b)}}}p_{\boldsymbol{D}(\boldsymbol{d})}\le\sum_{\substack{\delta(\boldsymbol{d})=b\\ \boldsymbol{d}\ne\boldsymbol{d}^{(b)}}}\frac{p_{D_b}p_n^{s-1}}{q^{g(b)}}\\ &\le M_s(b)\,\frac{p_{D_b}p_n^{s-1}}{q^{g(b)}}\le b^{\log_2\log_2 b}\,\frac{p_{D_b}p_n^{s-1}}{q^{g(b)}}.\end{aligned}\tag{5.11}$$

Combining both inequalities, the theorem follows. □

We may express Theorem 5.7 in terms of probabilities. Consider the set of all $\boldsymbol{f}\in\mathbb{P}^{\boldsymbol{D}(\boldsymbol{d})}(\mathbb{F}_q)$ when $\boldsymbol{d}$ runs through all the degree patterns with $\delta(\boldsymbol{d})=b$. If $\mathcal{P}_{\mathrm{irr}}^b$ denotes the probability for a uniformly random $\boldsymbol{f}$ to define an absolutely irreducible complete intersection $Z(\boldsymbol{f})\subset\mathbb{P}_{\mathbb{F}}^n$ of dimension $n-s$ and degree b, then Theorem 5.7 and (5.11) say that

$$\mathcal{P}_{\mathrm{irr}}^b\ge 1-\frac{1+14\,q^{-1}}{q^{n-s+3}}-\frac{2b^{\log_2\log_2 b}}{q^{g(b)}}$$

for $b>2$ or $n-s>3$.

6. Open questions

Several issues are left open in the context of this work.

- We have worked exclusively in the projective setting and it remains to adapt our approach to the affine case.
- The nonvanishing of our genericity polynomials is sufficient to guarantee the property that they work for. Can one find exact conditions for our properties that are necessary and sufficient? We have not even determined the dimensions of the sets of systems that violate the property.

- For a particular case of the previous question, see the remarks after (5.5). In that context, can one determine the dimension of the set of $\boldsymbol{f} \in \mathbb{P}_{\overline{K}}^{\boldsymbol{D}}$ not defining a normal, or absolutely irreducible, complete intersection of dimension $n - s$ and degree δ? Do both dimensions agree? Are they equidimensional subvarieties of $\mathbb{P}_{\overline{K}}^{\boldsymbol{D}}$?
- Elucidate the relation between the two models of varieties: systems of defining equations as in this paper, and Chow varieties. For example, unions of lines occur in the Chow point of view for curves in higher-dimensional spaces, but not in our considerations. More specifically: what is the dimension of the set of systems of s polynomials that define finite unions of linear spaces, each of codimension s? By Kumar (1990), such a union is a set-theoretic complete intersection if and only if it is connected (in the Zariski topology).
- Stephen Watt pointed out that one might investigate the genericity of computational "niceness" properties, such as a Gröbner basis computation in singly-exponential time.

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