

# INTERPOLATION BY DECOMPOSABLE UNIVARIATE POLYNOMIALS

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ABSTRACT. The usual univariate interpolation problem of finding a monic polynomial  $f$  of degree  $n$  that interpolates  $n$  given values is well understood. This paper studies a variant where  $f$  is required to be composite, say, a composition of two polynomials of degrees  $d$  and  $e$ , respectively, with  $de = n$ , and therefore  $d+e-1$  given values. Some special cases are easy to solve, and for the general case, we construct a homotopy between it and a special case. We compute a *geometric solution* of the algebraic curve presenting this homotopy, and this also provides an answer to the interpolation task. The computing time is polynomial in the geometric data, like the degree, of this curve. A consequence is that for almost all inputs, a decomposable interpolation polynomial exists.

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## 1. INTRODUCTION

For two univariate polynomials  $g, h \in \mathbb{F}[X]$  of degrees  $d, e$ , respectively, over a field  $\mathbb{F}$ , their *composition*

$$(1.1) \quad f = g(h) = g \circ h \in \mathbb{F}[X]$$

is a polynomial of degree  $n = de$ . If such  $g$  and  $h$  exist with degree at least 2, then  $f$  is called *decomposable* (or *composed*, *composite*, a *composition*). With a suitable normalization, explained in Section 3, we denote as  $C_{n,d}(\mathbb{F})$  the set of decomposable polynomials  $f$  as above.

The standard univariate interpolation problem is well understood. It asks for a monic interpolation polynomial  $f$ , given  $\deg f$  many constraints on values. When we require  $f \in C_{n,d}(\mathbb{F})$ , then only the  $d+e-1$  coefficients of  $g$  and  $h$  need to be determined. It is natural to impose the same number of constraints. We write  $\overline{\mathbb{F}}$  for an algebraic closure of  $\mathbb{F}$  and consider the following interpolation problem:

**Problem 1.1.** Given integers  $d, e \geq 2$ ,  $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{d+e-1}) \in \mathbb{F}^{d+e-1}$  with  $\alpha_i \neq \alpha_j$  for  $i \neq j$ ,  $\boldsymbol{\beta} = (\beta_1, \dots, \beta_{d+e-1}) \in \mathbb{F}^{d+e-1}$ , and  $n = de$ , does there exist  $f \in C_{n,d}(\overline{\mathbb{F}})$  such that  $f(\alpha_i) = \beta_i$  for  $1 \leq i \leq d+e-1$  and  $f = g \circ h$  with  $\deg g = d$  as above?

Standard univariate interpolation at  $n$  points  $(\alpha_i, \beta_i)$  has a solution if  $\alpha_i \neq \alpha_j$  for  $i \neq j$ , that is,  $\prod_{i \neq j} (\alpha_i - \alpha_j)$  is nonzero. But most polynomials are indecomposable: the decomposable ones form a subvariety of small dimension within the space of all polynomials (Fact 3.1). Thus interpolating at  $n$  points and then decomposing will usually not furnish a solution. Our parametrization provides a polynomial  $R$

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(Definition 3.9) such that  $R(\alpha) \neq 0$  guarantees a decomposable interpolant. Thus almost all such problems have a solution and Problem 1.1 a positive answer.

This issue arises, for example, in the black-box model of representing polynomials. If it is expensive to calculate any single value of the polynomial—say by some experiment—then one may wish to minimize the number of values sufficient to identify the polynomial. If, in addition, the polynomial is a priori known to be decomposable, then our method provides the optimal number of values required.

## 2. SPECIAL CASES OF THE INTERPOLATION PROBLEM

In this section, we show that Problem 1.1 has a positive answer for a generic choice of  $\alpha \in \mathbb{F}^{d+e-1}$  and certain, rather special,  $\beta \in \mathbb{F}^{d+e-1}$ . These results hold over an arbitrary field  $\mathbb{F}$ .

For any positive integer  $\ell$  and vector  $\mathbf{x} = (x_1, \dots, x_\ell) \in \mathbb{F}^\ell$ , we write  $\#\mathbf{x}$  for the number of different coordinates of  $\mathbf{x}$ , so that  $\#\mathbf{x} = \#\{x_1, \dots, x_\ell\}$ . Then  $\mathbf{x}$  defines a set partition  $\mathcal{P}_\mathbf{x}$  of  $\{1, \dots, \ell\}$  into  $\#\mathbf{x}$  blocks, where  $i, j \leq \ell$  belong to the same block if and only if  $x_i = x_j$ . The Stirling number of the second kind  $S_2(\ell, m)$  is the number of set partitions of  $\{1, \dots, \ell\}$  into exactly  $m$  blocks ( $m$ -partitions). For a polynomial  $f \in \mathbb{F}[X]$ , we write  $f(\mathbf{x}) = (f(x_1), \dots, f(x_\ell)) \in \mathbb{F}^\ell$ , so that  $\#f(\mathbf{x})$  equals the number of different values that  $f$  takes on  $\{x_1, \dots, x_\ell\}$ . For two set partitions  $\mathcal{P}$  and  $\mathcal{Q}$  we say that  $\mathcal{P}$  is a *refinement* of  $\mathcal{Q}$  and write  $\mathcal{P} \preceq \mathcal{Q}$ , if each block of  $\mathcal{P}$  is contained in one block of  $\mathcal{Q}$ . If  $\mathcal{P}$  consists of  $m$  blocks, we call it an  $m$ -refinement of  $\mathcal{Q}$ . Then  $m \geq \#\mathcal{Q}$ , and some  $m$ -refinement of  $\mathcal{Q}$  exists if and only if  $\ell \geq m \geq \#\mathcal{Q}$ . This induces a partial order on the set of partitions of  $\{1, \dots, \ell\}$ . The minimum of this lattice consists of  $\ell$  singletons, and its maximum of a single block. Let  $\alpha = (\alpha_1, \dots, \alpha_{d+e-1}) \in \mathbb{F}^{d+e-1}$  and  $(g, h)$  be a solution to Problem 1.1. Then

$$h(\alpha_i) = h(\alpha_j) \implies \beta_i = g \circ h(\alpha_i) = g \circ h(\alpha_j) = \beta_j \text{ for } i, j \leq d + e - 1.$$

Thus  $\mathcal{P}_{h(\alpha)} \preceq \mathcal{P}_\beta$  and  $\#h(\alpha) \geq \#\beta$ .

We use the polynomial  $R^{(1)}$  defined in (2.5) below.

**Lemma 2.1.** *The polynomial  $R^{(1)} \in \mathbb{F}[\mathbf{A}]$  is nonzero and has degree less than  $e^2 S_2(d+e-1, d-1)$ . For any  $\alpha = (\alpha_1, \dots, \alpha_{d+e-1}) \in \mathbb{F}^{d+e-1}$  with  $R^{(1)}(\alpha) \neq 0$  and any  $h = X^e + h_{e-1}X^{e-1} + \dots + h_1X \in \mathbb{F}[X]$  we have  $\#h(\alpha) \geq d$ .*

*Proof.* Assume that  $h = X^e + h_{e-1}X^{e-1} + \dots + h_1X \in \mathbb{F}[X]$  satisfies  $m = \#h(\alpha) < d$ . We consider the  $m$ -partition  $\mathcal{P} = \{P_1, \dots, P_m\} = \mathcal{P}_{h(\alpha)}$  of  $\{1, \dots, d+e-1\}$ . Thus  $i, j < d+e$  are in the same block of  $\mathcal{P}$  if and only if  $h(\alpha_i) = h(\alpha_j)$ . In order to facilitate notation, we sort the integers in each  $P_k \subseteq \{1, \dots, d+e-1\}$  in the natural order, let  $i_k \in P_k$  be the smallest element of  $P_k$ , and furthermore sort  $\{1, \dots, m\}$  according to the  $i_k$ 's, so that  $i_1 < i_2 < \dots < i_m$ . Then we have

$$(2.1) \quad h(\alpha_{i_k}) - h(\alpha_j) = 0 \text{ for all } k \leq m \text{ and } j \in P_k.$$

We also set  $P_k^{(1)} = P_k \setminus \{i_k\}$ . In particular, if  $P_k = \{i_k\}$  is a singleton, then  $P_k^{(1)} = \emptyset$ . The various instances of (2.1) provide

$$\sum_{1 \leq k \leq m} \#P_k^{(1)} = \sum_{1 \leq k \leq m} (\#P_k - 1) = \sum_{1 \leq k \leq m} \#P_k - m = d + e - 1 - m \geq e$$

linear equations on the coefficients of  $h$ .

We now pick  $e$  of these equations. We might take any  $e$  ones, but for a unique choice, we take the “first” ones, as follows. We let  $\ell \leq m$  be the smallest index so that

$$\sum_{1 \leq k \leq \ell} \#P_k^{(1)} \geq e, \quad P_\ell^{(1)} \neq \emptyset,$$

let  $P_\ell^{(2)} \subseteq P_\ell^{(1)}$  consist of the first  $e - \sum_{1 \leq k < \ell} \#P_k^{(1)}$  elements of  $P_\ell^{(1)}$ , and  $E = \bigcup_{1 \leq k < \ell} P_k^{(1)} \cup P_\ell^{(2)}$ . Then  $P_\ell^{(2)}$  is nonempty and  $\#E = e$ .

The equations (2.1) corresponding to  $E$  are:

$$(2.2) \quad \begin{cases} h(\alpha_{i_k}) - h(\alpha_j) = 0 & \text{for } 1 \leq k < \ell, j \in P_k^{(1)}, \\ h(\alpha_{i_\ell}) - h(\alpha_j) = 0 & \text{for } j \in P_\ell^{(2)}. \end{cases}$$

We denote by  $j_0$  the largest value of  $j \in P_\ell^{(2)}$ . Then these equations correspond to the following system of  $e$  linear equations for the coefficients of  $h$ :

$$(2.3) \quad \begin{pmatrix} \alpha_{i_1} - \alpha_j & \cdots & \alpha_{i_1}^e - \alpha_j^e \\ \vdots & & \vdots \\ \alpha_{i_\ell} - \alpha_{j_0} & \cdots & \alpha_{i_\ell}^e - \alpha_{j_0}^e \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ h_{e-1} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix},$$

where the indices correspond to those of (2.2). In particular,  $j$  has the same value on each line of the matrix, but these values are different for different lines.

If the coefficient matrix is nonsingular, then the system has no solution  $h$ , and thus our assumption about  $h$  is contradictory. We can reformulate this by replacing each occurrence of an  $\alpha_i$  in the matrix by an indeterminate  $A_i$  and letting

$$(2.4) \quad R_{\mathcal{P}} = \det \begin{pmatrix} A_{i_1} - A_j & \cdots & A_{i_1}^e - A_j^e \\ \vdots & & \vdots \\ A_{i_\ell} - A_{j_0} & \cdots & A_{i_\ell}^e - A_{j_0}^e \end{pmatrix} \in \mathbb{F}[\mathbf{A}],$$

where  $\mathbf{A} = (A_1, \dots, A_{d+e-1})$ . Then  $\#h(\boldsymbol{\alpha}) \geq d$  if  $R_{\mathcal{P}}(\boldsymbol{\alpha}) \neq 0$ .

We claim that the matrix for  $R_{\mathcal{P}}$  is nonsingular. For each of its rows, there is a value  $j$  so that  $A_j$  occurs on that line and nowhere else in the matrix, namely as  $A_j, \dots, A_j^e$ . In a Laplace expansion of the determinant, the cofactor of  $A_j^e$  is a matrix that also has this property. There is no cancellation for  $A_j^e$  and inductively, it follows that  $R_{\mathcal{P}} \neq 0$ . Its total degree is  $\sum_{1 \leq i \leq e} i = e(e+1)/2 < e^2$ .

The partition  $\mathcal{P}$  that gives rise to  $R_{\mathcal{P}}$  is an  $m$ -partition of  $\{1, \dots, d+e-1\}$ , for some  $m < d$ . We now describe a  $(d-1)$ -partition  $\mathcal{Q}$  of  $\{1, \dots, d+e-1\}$  with  $R_{\mathcal{P}} = R_{\mathcal{Q}}$ . We let  $E' = \bigcup_{1 \leq k < \ell} P_k \cup \{i_\ell\} \cup P_\ell^{(2)}$ . An index  $i \in \{1, \dots, d+e-1\}$  occurs somewhere in the matrix for  $R_{\mathcal{P}}$  if and only if  $i \in E'$ , and  $\#E' = e + \ell$ . Thus there are  $d+e-1 - (e + \ell) = d-1 - \ell$  such indices that do not occur in the matrix. Now  $\mathcal{Q}$  consists of the blocks  $P_1, \dots, P_{\ell-1}$  of  $\mathcal{P}$ ,  $\{i_\ell\} \cup P_\ell^{(2)}$ , and a singleton  $\{i\}$  for all  $i \in \{1, \dots, d+e-1\} \setminus E'$ . Thus  $\mathcal{Q}$  has  $\ell + d-1 - \ell = d-1$  blocks and is a  $(d-1)$ -partition of  $\{1, \dots, d+e-1\}$ . Furthermore, the indices in the added singletons occur neither in the matrix for  $R_{\mathcal{P}}$  nor in that for  $R_{\mathcal{Q}}$ , so that  $R_{\mathcal{P}} = R_{\mathcal{Q}}$ .

We set

$$(2.5) \quad R^{(1)} = \prod_{\mathcal{Q}} R_{\mathcal{Q}},$$

where the product runs over all  $(d-1)$ -partitions  $\mathcal{Q}$  of  $\{1, \dots, d+e-1\}$ . Then the claim of the lemma follows.  $\square$

For  $\alpha \in \mathbb{F}^{d+e-1}$ , let

$$\mathcal{C}_{\alpha} = \{h = X^e + h_{e-1}X^{e-1} + \dots + h_1X \in \mathbb{F}[X] : \#(h(\alpha)) \leq d\}.$$

We use the polynomial  $R^{(2)}$  defined in (2.7) below.

**Lemma 2.2.** *The polynomial  $R^{(2)} \in \mathbb{F}[\mathbf{A}]$  is nonzero and has degree less than  $e^2(S_2(d+e-1, d-1) + S_2(d+e-1, d))$ . For any  $\alpha \in \mathbb{F}^{d+e-1}$  with  $R^{(2)}(\alpha) \neq 0$ , we have*

$$\#\mathcal{C}_{\alpha} = S_2(d+e-1, d),$$

$\#h(\alpha) = d$  for all  $h \in \mathcal{C}_{\alpha}$ , and  $\alpha_i \neq \alpha_j$  whenever  $1 \leq i < j < d+e$ .

*Proof.* Let  $\alpha \in \mathbb{F}^{d+e-1}$  with  $R^{(1)}(\alpha) \neq 0$ , and  $h = X^e + h_{e-1}X^{e-1} + \dots + h_1X \in \mathcal{C}_{\alpha}$ . Lemma 2.1 implies that  $\#h(\alpha) = d$ .

We now follow the recipe for proving Lemma 2.1. Now  $\mathcal{P} = (P_1, \dots, P_d)$  is a  $d$ -partition of  $\{1, \dots, d+e-1\}$ , and all instances of (2.1) yield only

$$\sum_{1 \leq k \leq d} \#P_k^{(1)} = \sum_{1 \leq k \leq d} (\#P_k - 1) = \sum_{1 \leq k \leq d} \#P_k - d = d+e-1 - d = e-1$$

linear equations on the coefficients of  $h$ . Thus (2.3) would provide an  $(e-1) \times e$  system of linear equations. In order to obtain a square coefficient matrix, we move the quantities corresponding to the last column to the ‘‘constant’’ side and obtain:

$$(2.6) \quad \begin{pmatrix} \alpha_{i_1} - \alpha_j & \dots & \alpha_{i_1}^{e-1} - \alpha_j^{e-1} \\ \vdots & & \vdots \\ \alpha_{i_d} - \alpha_j & \dots & \alpha_{i_d}^{e-1} - \alpha_j^{e-1} \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ h_{e-1} \end{pmatrix} = \begin{pmatrix} \alpha_j^e - \alpha_{i_1}^e \\ \vdots \\ \alpha_j^e - \alpha_{i_d}^e \end{pmatrix},$$

where again the value of  $j$  is constant along each row of the matrix, but different for different rows.

We consider the matrix in  $\mathbb{F}[\mathbf{A}]^{(e-1) \times (e-1)}$ , where each entry  $\alpha_i$  in the coefficient matrix is replaced by the indeterminate  $A_i$  and let  $S_{\mathcal{P}} \in \mathbb{F}[\mathbf{A}]$  be its determinant. By the same argument as for  $R_{\mathcal{P}}$  in the previous proof, also  $S_{\mathcal{P}}$  is nonzero. Its degree is  $(e-1)e/2 < e^2$ . If  $S_{\mathcal{P}}(\alpha) \neq 0$ , then the coefficients of  $h$  are uniquely determined by (2.6).

An  $h \in \mathcal{C}_{\alpha}$  determines its partition  $\mathcal{P}$  uniquely via (2.1). On the other hand, if  $S_{\mathcal{P}}(\alpha) \neq 0$ , then (2.6) determines  $h$  uniquely. Thus we have a bijection between  $\mathcal{C}_{\alpha}$  and the set of  $d$ -partitions of  $\{1, \dots, d+e-1\}$ , both finite of size  $S_2(d+e-1, d)$ .

Recalling  $R^{(1)}$  from (2.5), we let

$$(2.7) \quad R^{(2)} = R^{(1)} \cdot \prod_{\mathcal{P}} S_{\mathcal{P}} \in \mathbb{F}[\mathbf{A}],$$

where the product runs over the set of  $d$ -partitions of  $\{1, \dots, d+e-1\}$ .

For the last claim, let  $1 \leq i < j < d+e$  and choose a  $d$ -partition  $\mathcal{P}$  of  $\{1, \dots, d+e-1\}$ , one of whose blocks, say  $P_k$ , is  $\{i, j\}$ . Since  $e \geq 2$ , there exists a  $(d-1)$ -partition on the other  $d+e-3 \geq d-1$  elements. In the matrix of indeterminates

defining  $S_{\mathcal{P}}$ ,  $A_i - A_j$  divides all entries in the row corresponding to  $P_k$ , and hence also  $S_{\mathcal{P}}$  and  $R^{(2)}$  are divisible by this difference. Thus  $\alpha_i \neq \alpha_j$  whenever  $R^{(2)}(\alpha) \neq 0$ . The claims of the lemma follow.  $\square$

The Bell number  $B_n$  counts the number of all set partitions of a set with  $n$  elements. The estimate in Berend & Tassa [3] yields

$$(2.8) \quad \begin{aligned} S_2(d+e-1, d-1) + S_2(d+e-1, d) &< B_{d+e-1} < \left(\frac{d+e-1}{\ln(d+e)}\right)^{d+e-1}, \\ \deg R^{(2)} &< e^2 B_{d+e-1} \leq (d+e)^{d+e-1}. \end{aligned}$$

For certain  $\beta$  with sufficiently few distinct entries, we can compute a composite interpolation polynomial  $f = g \circ h$  with  $\deg g = d$ , and the number of such polynomials.

**Proposition 2.3.** *Let  $\alpha \in \mathbb{F}^{d+e-1}$  satisfy  $R^{(2)}(\alpha) \neq 0$  and  $\#\beta \leq d$ . Then we can compute a solution to Problem 1.1 for  $(\alpha, \beta)$ . All solutions are defined over  $\mathbb{F}$  and the number of solutions equals the number of  $d$ -refinements of  $\mathcal{P}_{\beta}$ .*

*Proof.* We let  $\mathcal{P}$  be an arbitrary  $d$ -refinement of  $\mathcal{P}_{\beta}$ , that is,  $\mathcal{P} \preceq \mathcal{P}_{\beta}$ . Solving the nonsingular system (2.6) of linear equations corresponding to  $\mathcal{P}$  provides a solution  $(h_1, \dots, h_{e-1})$  with all  $h_i \in \mathbb{F}$ , and we set  $h = X^e + h_{e-1}X^{e-1} + \dots + h_1X \in \mathbb{F}[X]$ . Then  $\#h(\alpha) = d$  and for  $1 \leq k \leq d$ , we let  $i_k \in P_k \subset \{1, \dots, d+e-1\}$  be a representative of the  $k$ th block  $P_k$  of  $\mathcal{P}$ . The condition that  $g(h(\alpha_{i_k})) = \beta_{i_k}$  for  $k \leq d$  defines a unique monic interpolation polynomial  $g \in \mathbb{F}[X]$  of degree  $d$ , and  $(g, h)$  is a solution to Problem 1.1.

The proof of Lemma 2.2 notes a bijection between the set of  $h$  and that of the above partitions. This implies the claim on the number of solutions.  $\square$

**Example 2.4.** For  $d = e = 2$ , we consider the arbitrarily chosen  $\alpha = (5, 6, 7)$  and  $\beta = (3, 3, 3)$ . Then  $\mathcal{P}_{\beta} = \{1, 2, 3\}$  has the three 2-refinements  $\{\{1, 2\}, \{3\}\}$ ,  $\{\{1, 3\}, \{2\}\}$ , and  $\{\{2, 3\}, \{1\}\}$ . We find the corresponding three components  $g = x^2 + g_1x + g_0$  and  $h = x^2 + h_1x$  with  $(g_1, g_0, h_1)$  as

$$(58, 843, -11), (71, 1263, -12), (82, 1683, -13)$$

and their compositions  $f = g \circ h$  as

$$\begin{aligned} x^4 - 22x^3 + 179x^2 - 638x + 843, & x^4 - 24x^3 + 215x^2 - 852x + 1263, \\ x^4 - 26x^3 + 251x^2 - 1066x + 1683. & \end{aligned}$$

This works over any field  $\mathbb{F}$  with  $\text{char } \mathbb{F} \neq 2$ .  $\mathbb{F}_2$  does not have enough elements for such an example, but  $\mathbb{F}_4 = \mathbb{F}_2[y]/(y^2 + y + 1)$  does. For  $\alpha = (0, 1, y)$  and  $\beta = (y+1, y+1, y+1)$ , we find the following three coefficient vectors of components  $(g, h)$ :

$$(1, y+1, 1), (y+1, y+1, y), (y, y+1, y+1)$$

For all three, the composition is  $f = x^4 + x + y + 1$ . This is a *composition collision* of a type which is classified in Blankertz et al. [5].  $\square$

**Corollary 2.5.** *For an  $\alpha \in \mathbb{F}^{d+e-1}$  with  $R^{(2)} \neq 0$ , the set of solutions  $(g, h)$  to Problem 1.1 for  $(\alpha, \mathbf{0})$  is finite of cardinality  $S_2(d+e-1, d)$  and all solutions are defined over  $\mathbb{F}$ . For such a solution, we have  $\#h(\alpha) = d$ .*

*Proof.*  $\mathcal{P}_{\mathbf{0}}$  is the maximum partition in the lattice of partitions,  $\#\mathcal{P}_{\mathbf{0}} = 1$ , and it has  $S_2(d+e-1, d)$  many  $d$ -refinements.  $\square$

## 3. THE NUMBER OF INTERPOLANTS

We have shown with methods from linear algebra how to solve the interpolation problem for special  $\beta$ . The general case requires tools from algebraic geometry. We use notions like varieties and morphism only over algebraically closed fields, but they may be “defined over” some smaller field  $\mathbb{F}$  with algebraic closure  $\overline{\mathbb{F}}$ . Standard notions and notations of algebraic geometry can be found in, e.g., Kunz [25] or Shafarevich [30].

We now set some notation for the decomposable polynomials as in (1.1). We may assume all three polynomials to be monic (leading coefficient 1) and  $h$  original (constant coefficient 0, so that the graph contains the origin). All other compositions can be obtained from this special case by composing (on the left and on the right) with linear polynomials (polynomials of degree 1); see, e.g., von zur Gathen [13].

Thus we consider for a proper divisor  $d$  of  $n$  and  $e = n/d$

$$\begin{aligned}
 P_n(\mathbb{F}) &= \{f \in \mathbb{F}[X]: \deg f = n, f \text{ monic}\}, \\
 P_e^*(\mathbb{F}) &= \{h \in P_e(\mathbb{F}): h \text{ original}\}, \\
 (3.1) \quad \gamma_{n,d}: P_d(\mathbb{F}) \times P_e^*(\mathbb{F}) &\rightarrow P_n(\mathbb{F}) \text{ with } \gamma_{n,d}(g, h) = g \circ h, \\
 C_{n,d}(\mathbb{F}) &= \{f \in P_n(\mathbb{F}): \exists g, h \in P_d(\mathbb{F}) \times P_e^*(\mathbb{F}) \ f = g \circ h\} \\
 &= \text{im } \gamma_{n,d}, \\
 C_n(\mathbb{F}) &= \bigcup_{\substack{d|n \\ d \notin \{1, n\}}} C_{n,d}(\mathbb{F}).
 \end{aligned}$$

We may endow  $P_n(\mathbb{F})$  with a structure of  $n$ -dimensional vector space over  $\mathbb{F}$ , associating each  $f = X^n + f_{n-1}X^{e-1} + \dots + f_0$  with the vector  $(f_{n-1}, \dots, f_0)$ , and  $C_n(\mathbb{F})$  is the algebraic variety of decomposable polynomials. We drop the argument  $\mathbb{F}$  when it is clear from the context. When  $n$  is prime, then  $C_n(\mathbb{F})$  is empty, and in the following we always assume  $n$  to be composite. We have the following geometric description of  $C_{n,d}$ .

**Fact 3.1** (von zur Gathen & Matera [15, Theorem 2.2]). Let  $d$  be a proper divisor of  $n$  and  $\mathbb{F}$  be an algebraically closed field with  $\text{char}(\mathbb{F})$  not dividing  $d$ . Then  $C_{n,d}(\mathbb{F}) = \text{im } \gamma_{n,d}$  as in (3.1) is a closed irreducible algebraic subvariety of  $P_n(\mathbb{F})$  of dimension  $d + e - 1$  and degree at most  $d^{d+e-1}$ .

In all of the following, we assume that  $\text{char}(\mathbb{F})$  does not divide  $d$ . Our goal is to show that Problem 1.1 has an affirmative answer for a suitably generic instance  $(\alpha, \beta) \in \overline{\mathbb{F}}^{d+e-1} \times \overline{\mathbb{F}}^{d+e-1}$ , namely there exists an interpolation polynomial in  $C_{n,d}$ . For this purpose, we introduce the following incidence variety:

$$(3.2) \quad \Gamma_{n,d} = \{(g, h, \alpha, \beta) \in P_d(\overline{\mathbb{F}}) \times P_e^*(\overline{\mathbb{F}}) \times \overline{\mathbb{F}}^{d+e-1} \times \overline{\mathbb{F}}^{d+e-1} : g \circ h(\alpha) = \beta\}.$$

We have the following result.

**Lemma 3.2.**  $\Gamma_{n,d}$  is defined over  $\mathbb{F}$ , irreducible of dimension  $2(d+e-1)$  and degree at most  $n^{d+e-1}$ .

*Proof.* Let  $\mathbf{A} = (A_1, \dots, A_{d+e-1})$ ,  $\mathbf{B} = (B_1, \dots, B_{d+e-1})$ ,  $\mathbf{G} = (G_{d-1}, \dots, G_0)$  and  $\mathbf{H} = (H_{e-1}, \dots, H_1)$  be vectors of indeterminates over  $\overline{\mathbb{F}}$ . We denote by  $\mathbf{A}$  and  $\mathbf{B}$  the coordinates of  $\overline{\mathbb{F}}^{d+e-1} \times \overline{\mathbb{F}}^{d+e-1}$ , and by  $\mathbf{G}$  and  $\mathbf{H}$  the coordinates of

$P_d(\overline{\mathbb{F}}) \times P_e^*(\overline{\mathbb{F}})$ . Let  $G, H \in \mathbb{F}[\mathbf{G}, \mathbf{H}, X]$  be the polynomials

$$G = X^d + G_{d-1}X^{d-1} + \cdots + G_0, \quad H = X^e + H_{e-1}X^{e-1} + \cdots + H_1X,$$

and consider the  $\mathbb{F}$ -algebra morphism

$$\Phi: \mathbb{F}[\mathbf{A}, \mathbf{B}, \mathbf{G}, \mathbf{H}] \rightarrow \mathbb{F}[\mathbf{A}, \mathbf{G}, \mathbf{H}]$$

defined by  $\Phi(B_i) = (G \circ H)(A_i)$  for  $1 \leq i \leq d+e-1$ . It is clear that  $\Phi$  is surjective, and its kernel is

$$\ker \Phi = \langle B_1 - (G \circ H)(A_1), \dots, B_{d+e-1} - (G \circ H)(A_{d+e-1}) \rangle = \mathcal{I}(\Gamma_{n,d}),$$

where  $\mathcal{I}(\Gamma_{n,d}) \subset \mathbb{F}[\mathbf{A}, \mathbf{B}, \mathbf{G}, \mathbf{H}]$  denotes the ideal of polynomials vanishing on  $\Gamma_{n,d}$ . It follows that

$$\overline{\Phi}: \mathbb{F}[\mathbf{A}, \mathbf{B}, \mathbf{G}, \mathbf{H}]/\mathcal{I}(\Gamma_{n,d}) \rightarrow \mathbb{F}[\mathbf{A}, \mathbf{G}, \mathbf{H}]$$

is an isomorphism. Since  $\mathbb{F}[\mathbf{A}, \mathbf{G}, \mathbf{H}]$  is a domain,  $\Gamma_{n,d}$  is irreducible, and its dimension agrees with the Krull dimension of  $\mathbb{F}[\mathbf{A}, \mathbf{G}, \mathbf{H}]$ , namely  $2(d+e-1)$ .

Since  $\Gamma_{n,d}$  is defined by  $d+e-1$  equations of degree at most  $n = de$ , the Bézout inequality (see, e.g., Heintz [18] or Fulton [11]) implies the claimed upper bound on its degree.  $\square$

We have a natural projection

$$(3.3) \quad \pi: \Gamma_{n,d} \rightarrow \overline{\mathbb{F}}^{d+e-1} \times \overline{\mathbb{F}}^{d+e-1}.$$

In these terms, Problem 1.1 can be rephrased as follows:

**Problem 3.3.** Given  $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{d+e-1}) \in \overline{\mathbb{F}}^{d+e-1}$  with  $\alpha_i \neq \alpha_j$  for  $i \neq j$ , and  $\boldsymbol{\beta} = (\beta_1, \dots, \beta_{d+e-1}) \in \overline{\mathbb{F}}^{d+e-1}$ , is the fiber  $\pi^{-1}(\boldsymbol{\alpha}, \boldsymbol{\beta})$  nonempty?

Our next result shows that Problem 3.3 has a positive answer for most  $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \overline{\mathbb{F}}^{d+e-1} \times \overline{\mathbb{F}}^{d+e-1}$ .

**Corollary 3.4.** *The projection mapping  $\pi: \Gamma_{n,d} \rightarrow \overline{\mathbb{F}}^{d+e-1} \times \overline{\mathbb{F}}^{d+e-1}$  is dominant. In particular, if  $R^{(2)}(\boldsymbol{\alpha}) \neq 0$ , then the fiber  $\pi^{-1}(\boldsymbol{\alpha}, \boldsymbol{\beta})$  is of dimension zero.*

*Proof.* Lemma 3.2 shows that  $\Gamma_{n,d}$  is irreducible of dimension  $2(d+e-1)$ . Consider  $\pi: \Gamma_{n,d} \rightarrow \overline{\pi(\Gamma_{n,d})}$  and observe that both  $\Gamma_{n,d}$  and  $\overline{\pi(\Gamma_{n,d})}$  are irreducible varieties, where  $\overline{\pi(\Gamma_{n,d})}$  is the Zariski closure of  $\pi(\Gamma_{n,d})$  in  $\overline{\mathbb{F}}^{d+e-1} \times \overline{\mathbb{F}}^{d+e-1}$ . For an  $\boldsymbol{\alpha} \in \overline{\mathbb{F}}^{d+e-1}$  with  $R^{(2)}(\boldsymbol{\alpha}) \neq 0$ , the fiber  $\pi^{-1}(\boldsymbol{\alpha}, \mathbf{0})$  has dimension zero by Corollary 2.5. Therefore, the theorem on the dimension of fibers (see, e.g., [30, §6.3, Theorem 7]) shows that

$$2(d+e-1) - \dim \overline{\pi(\Gamma_{n,d})} = \dim \Gamma_{n,d} - \dim \overline{\pi(\Gamma_{n,d})} \leq 0.$$

It follows that  $2(d+e-1) \leq \dim \overline{\pi(\Gamma_{n,d})}$ , that is,  $\overline{\pi(\Gamma_{n,d})} = \overline{\mathbb{F}}^{d+e-1} \times \overline{\mathbb{F}}^{d+e-1}$ . This proves the first assertion. Finally, taking into account that  $\Gamma_{n,d}$  and  $\overline{\mathbb{F}}^{d+e-1} \times \overline{\mathbb{F}}^{d+e-1}$  are both irreducible varieties of dimension  $2(d+e-1)$ , the second assertion is a standard consequence of the theorem on the dimension of fibers.  $\square$

We may paraphrase the statement of Corollary 3.4 in terms of the interpolation problem in the following way: for any  $(\boldsymbol{\alpha}, \boldsymbol{\beta})$  with  $R^{(2)}(\boldsymbol{\alpha}) \neq 0$ , there is a finite positive number of solutions  $f = g \circ h \in C_{n,d}$  to the interpolation problem defined by  $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ . Now we discuss the possible number of interpolants.

Since the projection

$$\pi: \Gamma_{n,d} \rightarrow \overline{\mathbb{F}}^{d+e-1} \times \overline{\mathbb{F}}^{d+e-1}$$

is a dominant mapping defined over  $\mathbb{F}$ , it induces a finite field extension  $\mathbb{F}(\mathbf{A}, \mathbf{B}) \hookrightarrow \mathbb{F}(\Gamma_{n,d})$  between the fields of rational functions on  $\overline{\mathbb{F}}^{d+e-1} \times \overline{\mathbb{F}}^{d+e-1}$  and on  $\Gamma_{n,d}$  defined over  $\mathbb{F}$ . The degree of this field extension is the *degree of the morphism*  $\pi$  and is denoted by  $\deg \pi$ .

Assume that the field extension  $\mathbb{F}(\mathbf{A}, \mathbf{B}) \hookrightarrow \mathbb{F}(\Gamma_{n,d})$  is separable (in Proposition 3.7 below we prove this assertion). By, e.g., [18, Proposition 1],  $\#\pi^{-1}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \leq \deg \pi$  for any  $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \overline{\mathbb{F}}^{d+e-1} \times \overline{\mathbb{F}}^{d+e-1}$  with a finite fiber, with equality in a Zariski open subset of  $\overline{\mathbb{F}}^{d+e-1} \times \overline{\mathbb{F}}^{d+e-1}$ . In particular, from Corollary 2.5 we deduce that

$$\deg \pi \geq S_2(d+e-1, d).$$

We also have an easy upper bound.

**Lemma 3.5.** *We have  $\deg \pi \leq \deg C_{n,d} \leq d^{d+e-1}$ .*

*Proof.* Let  $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \overline{\mathbb{F}}^{d+e-1} \times \overline{\mathbb{F}}^{d+e-1}$  with  $\#\pi^{-1}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \deg \pi$ . Then  $\pi^{-1}(\boldsymbol{\alpha}, \boldsymbol{\beta})$  is isomorphic to the zero-dimensional variety

$$\{f \in C_{n,d} : f(\alpha_i) = \beta_i \text{ for } 1 \leq i \leq d+e-1\},$$

which is the intersection of  $C_{n,d}$  with an affine linear subspace of  $\overline{\mathbb{F}}^n$ . The Bézout inequality (see, e.g., [18] or [11]) implies the first inequality, and the second one follows from Theorem 3.1.  $\square$

Corollary 2.5 and Lemma 3.5 imply that

$$(3.4) \quad S_2(d+e-1, d) \leq \deg \pi \leq d^{d+e-1}.$$

To compare both sides of the inequality, we recall that  $d! S_2(d+e-1, d)$  is the number of surjective functions from  $\{1, \dots, d+e-1\}$  to  $\{1, \dots, d\}$  (see, e.g., Mezö [26, page 17]), while  $d^{d+e-1}$  is the total number of functions from  $\{1, \dots, d+e-1\}$  to  $\{1, \dots, d\}$ .

Next we want to show that  $\#\pi^{-1}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \deg \pi$  for most inputs. For this purpose, we have to understand the ramification locus of the projection  $\pi: \Gamma_{n,d} \rightarrow \overline{\mathbb{F}}^{d+e-1} \times \overline{\mathbb{F}}^{d+e-1}$  and the separability of the corresponding field extension. For  $(g, h, \boldsymbol{\alpha}, \boldsymbol{\beta}) \in \pi^{-1}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ ,  $\pi$  is *unramified* at  $(g, h, \boldsymbol{\alpha}, \boldsymbol{\beta})$  if the differential mapping  $d_{(g,h,\boldsymbol{\alpha},\boldsymbol{\beta})}\pi: T_{(g,h,\boldsymbol{\alpha},\boldsymbol{\beta})}\Gamma_{n,d} \rightarrow T_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(\overline{\mathbb{F}}^{d+e-1} \times \overline{\mathbb{F}}^{d+e-1})$  is injective.

Since the polynomials  $Q_i = (G \circ H)(A_i) - B_i$  with  $1 \leq i \leq d+e-1$  defining the incidence variety  $\Gamma_{n,d}$  generate a radical ideal, it follows that the tangent space  $T_{(g,h,\boldsymbol{\alpha},\boldsymbol{\beta})}$  of any  $(g, h, \boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Gamma_{n,d}$  is the kernel of the Jacobian matrix  $(\partial \mathbf{Q} / (\partial(\mathbf{G}, \mathbf{H}, \mathbf{A}, \mathbf{B}))(g, h, \boldsymbol{\alpha}, \boldsymbol{\beta}))$  of  $\mathbf{Q} = (Q_1, \dots, Q_{d+e-1})$  with respect to  $\mathbf{G}, \mathbf{H}, \mathbf{A}, \mathbf{B}$  at  $(g, h, \boldsymbol{\alpha}, \boldsymbol{\beta})$ . Since  $\pi$  is the linear mapping defined by the projection to  $\overline{\mathbb{F}}^{d+e-1} \times \overline{\mathbb{F}}^{d+e-1}$ , it is unramified at  $(g, h, \boldsymbol{\alpha}, \boldsymbol{\beta})$  if and only if the Jacobian matrix  $((\partial \mathbf{Q}) / (\partial(\mathbf{G}, \mathbf{H}))(g, h, \boldsymbol{\alpha}, \boldsymbol{\beta}))$  is of full rank.

Our first result considers the special case  $(\boldsymbol{\alpha}, \mathbf{0})$ .

**Lemma 3.6.** *For an  $\boldsymbol{\alpha}$  with  $R^{(2)}(\boldsymbol{\alpha}) \neq 0$ ,  $\pi$  is unramified at any point  $(g, h, \boldsymbol{\alpha}, \mathbf{0}) \in \pi^{-1}(\boldsymbol{\alpha}, \mathbf{0})$ .*



*Proof.* We let  $q = d + e - 1$  and consider the  $q \times q$  Jacobian matrix  $J = (\partial \mathbf{Q} / (\partial \mathbf{G}, \mathbf{H}))(g, h, \boldsymbol{\alpha}, \mathbf{0})$ :

$$(3.5) \quad J = \begin{pmatrix} h(\alpha_1)^{d-1} & \cdots & h(\alpha_1) & 1 & g'(h(\alpha_1))\alpha_1^{e-1} & \cdots & g'(h(\alpha_1))\alpha_1 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ h(\alpha_d)^{d-1} & \cdots & h(\alpha_d) & 1 & g'(h(\alpha_d))\alpha_d^{e-1} & \cdots & g'(h(\alpha_d))\alpha_d \\ h(\alpha_{d+1})^{d-1} & \cdots & h(\alpha_{d+1}) & 1 & g'(h(\alpha_{d+1}))\alpha_{d+1}^{e-1} & \cdots & g'(h(\alpha_{d+1}))\alpha_{d+1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ h(\alpha_q)^{d-1} & \cdots & h(\alpha_q) & 1 & g'(h(\alpha_q))\alpha_q^{e-1} & \cdots & g'(h(\alpha_q))\alpha_q \end{pmatrix}.$$

From Corollary 2.5 we have  $\#h(\boldsymbol{\alpha}) = d$  and by Lemma 2.2 that  $h(\alpha_i) \neq h(\alpha_j)$  for  $1 \leq i < j \leq d$ , while each of the remaining values  $h(\alpha_{d+j})$  agrees with an  $h(\alpha_{i_j})$  with  $1 \leq i_j \leq d$  for  $1 \leq j \leq e - 1$ . As a consequence, the upper-left  $d \times d$  submatrix  $J_1$  of  $J$  is invertible. Further, by subtracting the  $i_j$ th row from the  $(d + j)$ th row of  $J$  for  $1 \leq j \leq e - 1$ , we obtain a  $q \times q$  matrix with the same rank as  $J$  and of the following shape:

$$\left( \begin{array}{c|ccc} J_1 & & & * \\ \hline & g'(h(\alpha_{d+1}))(\alpha_{d+1}^{e-1} - \alpha_{i_1}^{e-1}) & \cdots & g'(h(\alpha_{d+1}))(\alpha_{d+1} - \alpha_{i_1}) \\ \mathbf{0} & \vdots & & \vdots \\ & g'(h(\alpha_q))(\alpha_q^{e-1} - \alpha_{i_{e-1}}^{e-1}) & \cdots & g'(h(\alpha_q))(\alpha_q - \alpha_{i_{e-1}}) \end{array} \right).$$

It follows that  $J$  is invertible if and only if this  $(e - 1) \times (e - 1)$  lower-right submatrix  $J_2$  is invertible.

Since  $g$  is monic polynomial of degree  $d$  vanishing on  $h(\boldsymbol{\alpha})$  and  $\#h(\boldsymbol{\alpha}) = d$ , any  $h(\alpha_i)$  is a simple root of  $g$ . This implies that  $g'(h(\alpha_{d+i})) \neq 0$  for  $1 \leq i \leq e - 1$ . We conclude that  $J_2$  is invertible if and only if the matrix

$$J_3 = \begin{pmatrix} \alpha_{d+1}^{e-1} - \alpha_{i_1}^{e-1} & \cdots & \alpha_{d+1} - \alpha_{i_1} \\ \vdots & & \vdots \\ \alpha_q^{e-1} - \alpha_{i_{e-1}}^{e-1} & \cdots & \alpha_q - \alpha_{i_{e-1}} \end{pmatrix}$$

is nonsingular. Up to row and column permutations and signs,  $J_3$  is the matrix in (2.6). (In the numbering of (2.6),  $d + j$  is in the same block of the partition as  $i_j$ .) Since  $R^{(2)}(\boldsymbol{\alpha}) \neq 0$ , it follows that  $J_3$  is nonsingular.  $\square$

In fact, the lemma's statement and proof are valid for any  $\boldsymbol{\beta}$  with  $\#\boldsymbol{\beta} \leq d$ , not just  $\boldsymbol{\beta} = \mathbf{0}$ , using Proposition 2.3 instead of Corollary 2.5.

Next we show that the field extension  $\mathbb{F}(\mathbf{A}, \mathbf{B}) \hookrightarrow \mathbb{F}(\Gamma_{n,d})$  is separable. This will show that  $\#\pi^{-1}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \deg \pi$  for any suitably generic  $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ .

**Proposition 3.7.** *The field extension  $\mathbb{F}(\mathbf{A}, \mathbf{B}) \hookrightarrow \mathbb{F}(\Gamma_{n,d})$  is separable.*

*Proof.* The Jacobian  $\mathbf{J} = \partial \mathbf{Q} / \partial (\mathbf{G}, \mathbf{H}) \in \mathbb{F}[\mathbf{G}, \mathbf{H}, \mathbf{A}]^{q \times q}$  is the matrix in (3.5) with  $g, h, \alpha$  replaced by vectors of indeterminates  $\mathbf{G}, \mathbf{H}, \mathbf{A}$ . Lemma 3.6 shows that  $\det \mathbf{J}$  does not vanish at certain points, so in particular  $\det \mathbf{J} \neq 0$  as a polynomial.

In the proof of Lemma 3.2 it is shown that the  $\mathbb{F}$ -algebra morphism  $\overline{\Phi}: \mathbb{F}[\Gamma_{n,d}] \rightarrow \mathbb{F}[\mathbf{A}, \mathbf{G}, \mathbf{H}]$ , defined by  $\overline{\Phi}(\overline{B}_j) = (G \circ H)(A_j)$  for  $1 \leq j \leq d + e - 1$ , where  $\overline{B}_j \in \mathbb{F}[\Gamma_{n,d}] = \mathbb{F}[\mathbf{A}, \mathbf{B}, \mathbf{G}, \mathbf{H}] / \mathcal{I}(\Gamma_{n,d})$  is the coordinate function of  $\Gamma_{n,d}$  defined

by  $B_j$ , is an isomorphism. As  $\det \mathbf{J}$  is a nonzero element of  $\mathbb{F}[\mathbf{A}, \mathbf{G}, \mathbf{H}]$ , we have  $\overline{\Phi}^{-1}(\det \mathbf{J}) \neq 0$ , and the fact that  $\mathbb{F}[\Gamma_{n,d}]$  is a domain implies that  $\overline{\Phi}^{-1}(\det \mathbf{J})$  is not a zero divisor of  $\mathbb{F}[\Gamma_{n,d}]$ . As a consequence,  $\Gamma_{n,d} \cap \{\det \mathbf{J} \neq 0\}$  is a nonempty Zariski open subset of  $\Gamma_{n,d}$ . In particular, the projection  $\pi : \Gamma_{n,d} \rightarrow \overline{\mathbb{F}}^{d+e-1} \times \overline{\mathbb{F}}^{d+e-1}$  is unramified at any  $(\mathbf{g}, \mathbf{h}, \boldsymbol{\alpha}, \boldsymbol{\beta})$  in this set.

According to Iversen [21, Proposition T.8], the extension  $\mathbb{F}(\mathbf{A}, \mathbf{B}) \hookrightarrow \mathbb{F}(\Gamma_{n,d})$  is separable if and only if  $\pi : \Gamma_{n,d} \rightarrow \overline{\mathbb{F}}^{d+e-1} \times \overline{\mathbb{F}}^{d+e-1}$  is unramified in a nonempty Zariski open subset of  $\Gamma_{n,d}$ , which proves the proposition.  $\square$

Now [18, Proposition 1] shows that  $\deg \pi = \max \#\pi^{-1}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ , where the maximum is taken among all fibers with finite cardinality, and there exists a nonempty Zariski open subset of  $\overline{\mathbb{F}}^{d+e-1} \times \overline{\mathbb{F}}^{d+e-1}$  such that  $\#\pi^{-1}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \deg \pi$  for all its elements. Schost [29, Proposition 3] and its proof allow to describe more precisely an open subset for the condition that  $\#\pi^{-1}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \deg \pi$ .

**Fact 3.8.** There exists a nonzero polynomial  $R^{(3)} \in \mathbb{F}[\mathbf{A}, \mathbf{B}]$  of total degree at most  $2 \deg \pi \deg \Gamma_{n,d}$  with the following property: for any  $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \overline{\mathbb{F}}^{d+e-1} \times \overline{\mathbb{F}}^{d+e-1}$  with  $R^{(3)}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \neq 0$ , we have  $\#\pi^{-1}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \deg \pi$ .

We have introduced two polynomial conditions on  $\boldsymbol{\alpha}$ :  $R^{(2)}$  in (2.7) and  $R^{(3)}$  in the previous fact. We now combine them into a single polynomial:

$$(3.6) \quad R = R^{(2)}R^{(3)}.$$

Then  $R \in \mathbb{F}[\mathbf{A}]$  is nonzero and from (2.8), (3.4), and Lemma 3.2, we have

$$(3.7) \quad \deg R < (d+e)^{d+e-1} + 2(d^2e)^{d+e-1} \leq 3(d^2e)^{d+e-1}.$$

**Definition 3.9.** For  $d, e \geq 2$  and a field  $\mathbb{F}$ , we call  $\boldsymbol{\alpha} \in \overline{\mathbb{F}}^{d+e-1}$  *generic* if  $R(\boldsymbol{\alpha}) \neq 0$ , and the nonempty Zariski-open set  $\mathcal{U} = \{R \neq 0\} \subset \overline{\mathbb{F}}^{d+e-1}$  consists of these generic  $\boldsymbol{\alpha}$ . For any  $\boldsymbol{\beta} \in \overline{\mathbb{F}}^{d+e-1}$ , we also call  $(\boldsymbol{\alpha}, \boldsymbol{\beta})$  *generic*.

We can now give a criterion under which  $\pi$  is unramified.

**Lemma 3.10.** *Let  $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \overline{\mathbb{F}}^{d+e-1} \times \overline{\mathbb{F}}^{d+e-1}$  be generic. Then  $\#\pi^{-1}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \deg \pi$  and  $\pi$  is unramified at all  $(g, h, \boldsymbol{\alpha}, \boldsymbol{\beta}) \in \pi^{-1}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ .*

*Proof.* From Fact 3.8, we know that  $\#\pi^{-1}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \deg \pi$ . Thus the restriction  $\pi|_{\pi^{-1}(\mathcal{U})} : \pi^{-1}(\mathcal{U}) \rightarrow \mathcal{U}$  is a finite morphism. Then for any  $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{U}$ , the Principle of conservation of number (see, e.g., Danilov [9, §5.7]) asserts that

$$\deg \pi = \sum_{(g,h,\boldsymbol{\alpha},\boldsymbol{\beta}) \in \pi^{-1}(\boldsymbol{\alpha},\boldsymbol{\beta})} \deg_{(g,h,\boldsymbol{\alpha},\boldsymbol{\beta})} \pi,$$

where

$$\deg_{(g,h,\boldsymbol{\alpha},\boldsymbol{\beta})} \pi = \dim_{\mathbb{F}}(\mathcal{O}_{\Gamma_{n,d},(g,h,\boldsymbol{\alpha},\boldsymbol{\beta})} / \pi^*(\mathfrak{m}_{(\boldsymbol{\alpha},\boldsymbol{\beta})}) \mathcal{O}_{\Gamma_{n,d},(g,h,\boldsymbol{\alpha},\boldsymbol{\beta})}),$$

$\mathcal{O}_{\Gamma_{n,d},(g,h,\boldsymbol{\alpha},\boldsymbol{\beta})}$  is the local ring of rational functions on  $\Gamma_{n,d}$  defined over  $\mathbb{F}$  which are well-defined at  $(g, h, \boldsymbol{\alpha}, \boldsymbol{\beta})$  and  $\mathfrak{m}_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$  is the maximal ideal of rational functions on  $\overline{\mathbb{F}}^{d+e-1} \times \overline{\mathbb{F}}^{d+e-1}$ , defined over  $\mathbb{F}$  and vanishing at  $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ .

Now the fact that  $\#\pi^{-1}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \deg \pi$  implies that  $\deg_{(g,h,\boldsymbol{\alpha},\boldsymbol{\beta})} \pi = 1$  for any  $(g, h, \boldsymbol{\alpha}, \boldsymbol{\beta}) \in \pi^{-1}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ . This means that  $\mathcal{O}_{\Gamma_{n,d},(g,h,\boldsymbol{\alpha},\boldsymbol{\beta})} / \pi^*(\mathfrak{m}_{(\boldsymbol{\alpha},\boldsymbol{\beta})}) \mathcal{O}_{\Gamma_{n,d},(g,h,\boldsymbol{\alpha},\boldsymbol{\beta})}$  is isomorphic to  $\mathbb{F}$ . It follows that

$$\pi^*(\mathfrak{m}_{(\boldsymbol{\alpha},\boldsymbol{\beta})}) = \mathfrak{m}_{(g,h,\boldsymbol{\alpha},\boldsymbol{\beta})}$$

for all the points  $(g, h, \boldsymbol{\alpha}, \boldsymbol{\beta}) \in \pi^{-1}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ , which by, e.g., [9, §5.2], implies that  $\pi$  is unramified at all such points.  $\square$

#### 4. HOMOTOPY FOR INTERPOLATION

Our general task is to find composite interpolating polynomials for given generic  $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathbb{F}^{d+e-1} \times \mathbb{F}^{d+e-1}$ , where  $\text{char}(\mathbb{F})$  does not divide  $d$ . Corollary 2.5 provides a solution for the special value  $\boldsymbol{\beta} = \mathbf{0}$ . Our main tool is a homotopy connecting these two instances.

Recall that the interpolants for  $(\boldsymbol{\alpha}, \boldsymbol{\beta}) = (\alpha_1, \dots, \alpha_{d+e-1}, \beta_1, \dots, \beta_{d+e-1})$  are the common zeros  $(g, h)$  of the polynomial system

$$(4.1) \quad (G \circ H)(\alpha_i) - \beta_i = 0 \quad \text{for } 1 \leq i \leq d+e-1,$$

where  $G = X^d + G_{d-1}X^{d-1} + \dots + G_0$  and  $H = X^e + H_{e-1}X^{e-1} + \dots + H_1X$  are univariate polynomials in  $X$  with  $d+e-1$  indeterminate coefficients  $G_{d-1}, \dots, G_0$  and  $H_{e-1}, \dots, H_1$ . The deformation describing the homotopy is the variety  $V^* \subset \overline{\mathbb{F}}^{d+e}$ , defined over  $\mathbb{F}$  and determined by the equations

$$(4.2) \quad (G \circ H)(\alpha_i) - \beta_i S = 0 \quad \text{for } 1 \leq i \leq d+e-1.$$

Here,  $S$  is a parameter whose values range over  $\overline{\mathbb{F}}$ . For  $S = 0$ , we have the special problem already solved, and  $S = 1$  is our general task. More precisely, we shall be interested in a curve  $V \subset V^*$  which consists of certain irreducible components over  $\mathbb{F}$  of the variety  $V^*$ . Since each equation in (4.2) has degree at most  $d$ , the Bézout inequality then implies that

$$(4.3) \quad \delta = \deg V \leq \deg V^* \leq d^{d+e-1}.$$

The algorithm solving system (4.1) may be divided into three main parts. First, we compute a bivariate polynomial  $m_i(S, T)$  defining an irreducible plane curve birationally equivalent to an irreducible component of  $V$ . Then we extend the computation of the polynomial  $m_i(S, T)$  to the computation of the birational inverse itself. Finally, we substitute 1 for  $S$  in the birational inverse computed in the previous step and recover at least one solution to system (4.1). We use a representation of varieties called a *geometric solution* or a *rational univariate representation* which is well suited for algorithmic purposes (see, e.g., Rouillier [28], Giusti et al. [17]).

**4.1. Geometric solutions.** Let  $X_1, \dots, X_n$  be indeterminates over  $\mathbb{F}$  and denote  $\mathbf{X} = (X_1, \dots, X_n)$ . We start with the definition of a geometric solution of a zero-dimensional variety defined over  $\mathbb{F}$ . Let  $V = \{\mathbf{P}_1, \dots, \mathbf{P}_D\} \subset \overline{\mathbb{F}}^n$  be a zero-dimensional reduced variety defined over  $\mathbb{F}$ , and  $\mathcal{L} \in \mathbb{F}[\mathbf{X}]$  a linear form which separates the points of  $V$ , that is,  $\mathcal{L}(\mathbf{P}_i) \neq \mathcal{L}(\mathbf{P}_j)$  for  $i \neq j$ . A *geometric solution* of  $V$  consists of

- $\boldsymbol{\lambda} \in \mathbb{F}^n$  so that the linear form  $\mathcal{L} = \boldsymbol{\lambda} \cdot \mathbf{X} = \lambda_1 X_1 + \dots + \lambda_n X_n \in \mathbb{F}[\mathbf{X}]$  separates the points of  $V$ ,
- the minimal polynomial  $m_{\boldsymbol{\lambda}} = \prod_{1 \leq i \leq D} (T - \mathcal{L}(\mathbf{P}_i)) \in \mathbb{F}[T]$  of  $\mathcal{L}$  in  $V$ , where  $T$  is a new variable,
- polynomials  $w_1, \dots, w_n \in \mathbb{F}[T]$  with  $\deg w_j < D$  for  $1 \leq j \leq n$  such that

$$V = \{(w_1(\eta), \dots, w_n(\eta)) \in \overline{\mathbb{F}}^n : \eta \in \overline{\mathbb{F}}, m_{\boldsymbol{\lambda}}(\eta) = 0\}.$$

Thus  $\mathbf{P}_i = (w_1(\mathcal{L}(\mathbf{P}_i)), \dots, w_n(\mathcal{L}(\mathbf{P}_i)))$  for  $1 \leq i \leq D$ , and each  $w_j$  interpolates on  $D$  pairwise distinct points  $\mathcal{L}(\mathbf{P}_1), \dots, \mathcal{L}(\mathbf{P}_D)$ .

Next, we consider this notion for equidimensional varieties of dimension one, that is, for curves, defined over  $\mathbb{F}$ . Let  $n \geq 2$ ,  $V \subset \overline{\mathbb{F}}^n$  be a reduced variety defined over  $\mathbb{F}$  of pure dimension 1 and degree  $\delta$ , and let  $V = V_1 \cup \dots \cup V_t$  be the decomposition over  $\mathbb{F}$  of  $V$  into irreducible components. Suppose that the indeterminate  $X_1$  forms a separable transcendence basis of all field extensions  $\mathbb{F} \hookrightarrow \mathbb{F}(V_i)$  for  $1 \leq i \leq t$ , that is, each  $\mathbb{F}(X_1) \hookrightarrow \mathbb{F}(V_i)$  is a finite separable field extension. Denote by  $D_i$  its degree and let  $D = \sum_{1 \leq i \leq t} D_i$ . In particular, the linear projection  $\pi : V_i \rightarrow \overline{\mathbb{F}}^1$  defined by  $\pi(\mathbf{x}) = x_1$  is a dominant morphism of degree  $D_i$  for  $1 \leq i \leq t$ . The behavior of the degree of a variety under linear maps (see, e.g., [18, Lemma 2]) implies that

$$(4.4) \quad D = \sum_{1 \leq i \leq t} D_i \leq \sum_{1 \leq i \leq t} \deg V_i = \delta.$$

Let  $\mathbf{\Lambda} = (\Lambda_2, \dots, \Lambda_n)$  be a vector of new indeterminates. The extension  $V_{\mathbb{F}(\mathbf{\Lambda})}$  of  $V$  to  $\overline{\mathbb{F}(\mathbf{\Lambda})}^n$  is a variety of pure dimension 1 and its coordinate ring is isomorphic to  $\mathbb{F}(\mathbf{\Lambda}) \otimes_{\mathbb{F}} \mathbb{F}[V]$ . Consider the generic linear form

$$(4.5) \quad \mathcal{L}_{\mathbf{\Lambda}} = \mathbf{\Lambda} \cdot \mathbf{X}^* = \Lambda_2 X_2 + \dots + \Lambda_n X_n.$$

As the mapping  $\mathbb{F}[V] \rightarrow \mathbb{F}[V_1] \times \dots \times \mathbb{F}[V_t]$  defined by  $\xi \mapsto (\xi|_{V_1}, \dots, \xi|_{V_t})$  is an isomorphism of  $\mathbb{F}$ -algebras, we identify  $\mathbb{F}[V]$  with  $\mathbb{F}[V_1] \times \dots \times \mathbb{F}[V_t]$ . For  $\xi \in \mathbb{F}[V]$ , we denote its restriction to  $V_i$  by  $\xi^{(i)}$  for  $1 \leq i \leq t$ . Let  $\xi_1, \dots, \xi_n \in \mathbb{F}[V]$  be the coordinate functions of  $V$  induced by  $X_1, \dots, X_n$ , and set  $\widehat{\mathcal{L}} = \mathcal{L}_{\mathbf{\Lambda}}(\xi_2, \dots, \xi_n) \in \mathbb{F}(\mathbf{\Lambda}) \otimes_{\mathbb{F}} \mathbb{F}[V]$ . Since  $\xi_1^{(i)}$  and  $\widehat{\mathcal{L}}^{(i)}$  are algebraically dependent over  $\mathbb{F}(\mathbf{\Lambda})$ , there exists an irreducible polynomial  $m_{\mathbf{\Lambda}}^{(i)} \in \mathbb{F}[\mathbf{\Lambda}, X_1, T]$ , separable with respect to  $T$ , such that  $m_{\mathbf{\Lambda}}^{(i)}(\mathbf{\Lambda}, \xi_1^{(i)}, \widehat{\mathcal{L}}^{(i)}) = 0$  holds in  $\mathbb{F}(\mathbf{\Lambda}) \otimes_{\mathbb{F}} \mathbb{F}[V_i]$  for  $1 \leq i \leq t$ . These polynomials are pairwise relatively prime, and defining

$$m_{\mathbf{\Lambda}} = \prod_{1 \leq i \leq t} m_{\mathbf{\Lambda}}^{(i)},$$

we find the following identity in  $\mathbb{F}(\mathbf{\Lambda}) \otimes_{\mathbb{F}} \mathbb{F}[V]$ :

$$(4.6) \quad m_{\mathbf{\Lambda}}(\mathbf{\Lambda}, \xi_1, \widehat{\mathcal{L}}) = 0.$$

From, e.g., [29, Proposition 1], we deduce the following bounds:

- $\deg_T m_{\mathbf{\Lambda}} = D$ ,
- $\deg_{X_1} m_{\mathbf{\Lambda}} \leq \delta$ ,
- $\deg_{\mathbf{\Lambda}} m_{\mathbf{\Lambda}} \leq \delta$ .

Taking partial derivatives with respect to  $\Lambda_j$  in (4.6), we deduce the identity

$$(4.7) \quad \frac{\partial m_{\mathbf{\Lambda}}}{\partial \Lambda_j}(\mathbf{\Lambda}, \xi_1, \widehat{\mathcal{L}}) + \xi_j \frac{\partial m_{\mathbf{\Lambda}}}{\partial T}(\mathbf{\Lambda}, \xi_1, \widehat{\mathcal{L}}) = 0$$

in  $\mathbb{F}(\mathbf{\Lambda}) \otimes_{\mathbb{F}} \mathbb{F}[V]$  for  $2 \leq j \leq n$ . The polynomial  $\partial m_{\mathbf{\Lambda}} / \partial T$  is nonzero, and the resultant  $\text{Res}_T(m_{\mathbf{\Lambda}}, \partial m_{\mathbf{\Lambda}} / \partial T)$  is also nonzero (see, e.g., Gímenez & Matera [16, Lemma 3.2]).

We take some  $\boldsymbol{\lambda} \in \mathbb{F}^{n-1}$  such that  $\deg_T m_{\mathbf{\Lambda}}(\boldsymbol{\lambda}, X_1, T) = D$  and  $\text{Res}_T(m_{\mathbf{\Lambda}}, \partial m_{\mathbf{\Lambda}} / \partial T)(\boldsymbol{\lambda}, X_1) \neq 0$ . Then the linear form  $\mathcal{L} = \boldsymbol{\lambda} \cdot \mathbf{X}^*$  with  $\mathbf{X}^* = (X_2, \dots, X_n)$  induces a primitive element of the separable field extension  $\mathbb{F}(X_1) \hookrightarrow \mathbb{F}(V_i)$  for  $1 \leq i \leq t$ , and the minimal polynomials of  $\mathcal{L}$  in the field extensions

$\mathbb{F}(X_1) \hookrightarrow \mathbb{F}(V_i)$  are pairwise relatively prime (see, e.g., [16, Proposition 3.4]). If  $\ell$  is the coordinate function of  $\mathbb{F}[V]$  defined by  $\mathcal{L}$ , we then say that  $\ell$  is a *primitive element* of the extension  $\mathbb{F}(X_1) \hookrightarrow \mathbb{F}(V)$ , where  $\mathbb{F}(V)$  is total ring of fractions of  $\mathbb{F}[V]$ . Since  $\deg_T m_\Lambda = D$ , it follows that  $m_\Lambda(\boldsymbol{\lambda}, X_1, T)$  is the minimal polynomial of  $\ell$  in the extension  $\mathbb{F}(X_1) \hookrightarrow \mathbb{F}(V)$ . Setting

$$(4.8) \quad m_\lambda(X_1, T) = m_\Lambda(\boldsymbol{\lambda}, X_1, T), \quad v_j(X_1, T) = -\frac{\partial m_\Lambda}{\partial \Lambda_j}(\boldsymbol{\lambda}, X_1, T) \text{ in } \mathbb{F}[X_1, T]$$

and substituting  $\boldsymbol{\lambda}$  for  $\boldsymbol{\Lambda}$  in (4.6)–(4.7), we obtain the following identities of  $\mathbb{F}[V]$ :

$$(4.9) \quad m_\lambda(\xi_1, \ell) = 0, \quad \frac{\partial m_\lambda}{\partial T}(\xi_1, \ell)\xi_j - v_j(\xi_1, \ell) = 0 \quad (2 \leq j \leq n).$$

Thus the polynomials

$$m_\lambda(X_1, \mathcal{L}) \text{ and } \frac{\partial m_\lambda}{\partial T}(X_1, \mathcal{L})X_j - v_j(X_1, \mathcal{L}) \quad (2 \leq j \leq n)$$

belong to  $\mathcal{I}(V) \subset \mathbb{F}[X_1, \dots, X_n]$ . They satisfy the bounds  $\deg_{X_1} v_j \leq \delta$  and  $\deg_T v_j \leq D$  for  $2 \leq j \leq n$ . Finally, the equations

$$m_\lambda(X_1, \mathcal{L}) = 0, \quad \frac{\partial m_\lambda}{\partial T}(X_1, \mathcal{L})X_j - v_j(X_1, \mathcal{L}) = 0 \quad (2 \leq j \leq n),$$

constitute a system of equations for the variety  $V$  in the Zariski dense open subset  $V \cap \{(\partial m_\lambda / \partial T)(X_1, \mathcal{L}) \neq 0\}$  of  $V$ . This motivates the following definition.

**Definition 4.1.** With assumptions as above, a *geometric solution* of  $V$  consists of the following items:

- a linear form  $\mathcal{L} = \boldsymbol{\lambda} \cdot \mathbf{X}^* \in \mathbb{F}[\mathbf{X}^*]$  which induces a primitive element  $\ell$  of the extension  $\mathbb{F}(X_1) \hookrightarrow \mathbb{F}(V)$ ,
- the minimal polynomial  $m_\lambda \in \mathbb{F}[X_1][T]$  of  $\ell$ ,
- a generic *parametrization*  $v_2, \dots, v_n$  of  $V$  by the zeros of  $m_\lambda$ , of the form

$$\frac{\partial m_\lambda}{\partial T}(X_1, T)X_j - v_j(X_1, T) \quad (2 \leq j \leq n),$$

with  $v_j \in \mathbb{F}[X_1][T]$ ,  $\deg_T v_j < D$ ,  $\deg_{X_1} v_j \leq \delta$  and  $(\partial m_\lambda / \partial T)(X_1, \mathcal{L})X_j - v_j(X_1, \mathcal{L}) \in \mathcal{I}(V)$  for  $j \leq n$ .

We observe that the polynomial  $m_\lambda \in \mathbb{F}[X_1, T]$  can be also defined as follows: consider the linear map

$$\pi_\lambda: V \rightarrow \overline{\mathbb{F}}^2, \quad \pi_\lambda(\mathbf{x}) = (x_1, \boldsymbol{\lambda} \cdot \mathbf{x}^*).$$

The Zariski closure of  $\pi_\lambda(V)$  is a plane curve of degree at most  $\delta$ , which is indeed defined by  $m_\lambda(X_1, T) = 0$ . Further, the projection  $\pi_\lambda$  constitutes a birational equivalence between  $V$  and the plane curve  $\mathcal{C} = \{m_\lambda(X_1, T) = 0\}$ , namely there is a rational inverse of  $\pi_\lambda$ . More precisely, let

$$V_{\text{deg}} = \{\mathbf{x} \in \overline{\mathbb{F}}^n : (\partial m_\lambda / \partial T)(x_1, \mathcal{L}(\mathbf{x})) = 0\},$$

$$\mathcal{C}_{\text{deg}} = \{(x_1, t) \in \overline{\mathbb{F}}^2 : (\partial m_\lambda / \partial T)(x_1, t) = 0\}.$$

The fact that  $\text{Res}_T(m_\lambda, \partial m_\lambda / \partial T) \neq 0$  implies that  $V \setminus V_{\text{deg}}$  and  $\mathcal{C} \setminus \mathcal{C}_{\text{deg}}$  are Zariski dense open subsets of  $V$  and  $\mathcal{C}$  respectively. We have the following result.

**Fact 4.2** (Cafure & Matera [7, Proposition 6.3]).  $\pi_\lambda|_{V \setminus V_{\text{deg}}} : V \setminus V_{\text{deg}} \rightarrow \mathcal{C} \setminus \mathcal{C}_{\text{deg}}$  is an isomorphism of open sets defined over  $\mathbb{F}$ .

We remark that the inverse of  $\pi_{\lambda}|_{V \setminus V_{\text{deg}}}$  is defined in the following way:

$$\begin{aligned} \mathcal{C} \setminus \mathcal{C}_{\text{deg}} &\rightarrow V \setminus V_{\text{deg}}, \\ (\mathbf{x}, t) &\mapsto \left( x_1, \frac{v_2(x_1, t)}{(\partial m_{\lambda} / \partial T)(x_1, t)}, \dots, \frac{v_n(x_1, t)}{(\partial m_{\lambda} / \partial T)(x_1, t)} \right). \end{aligned}$$

**4.2. Power series expansions.** Let  $n \geq 2$ ,  $V \subset \overline{\mathbb{F}}^n$  be a reduced variety defined over  $\mathbb{F}$  of pure dimension 1 and degree  $\delta$ , and let  $V = V_1 \cup \dots \cup V_t$  be the decomposition over  $\mathbb{F}$  of  $V$  into irreducible components. Let  $\mathcal{L} = \lambda \cdot \mathbf{X}^* \in \mathbb{F}[\mathbf{X}^*]$  be a linear form and  $m_{\lambda}, v_2, \dots, v_n \in \mathbb{F}[X_1][T]$  which form a geometric solution of  $V$  as in the previous subsection.

$V$  is birationally equivalent to the plane curve  $\mathcal{C} \subset \overline{\mathbb{F}}^2$  defined by  $m_{\lambda}$ . The fact that  $V = V_1 \cup \dots \cup V_t$ , where each  $V_i$  is irreducible of dimension 1, implies  $\mathcal{C}$  has a decomposition  $\mathcal{C} = \mathcal{C}_1 \cup \dots \cup \mathcal{C}_t$ , where each  $\mathcal{C}_i$  is defined by an irreducible factor  $m_i$  of  $m_{\lambda}$ , so that the irreducible factorization of  $m_{\lambda}$  in  $\mathbb{F}[X_1, T]$  is of the form

$$m_{\lambda}(X_1, T) = \prod_{i=1}^t m_i(X_1, T).$$

As the linear form  $\mathcal{L}$  is a primitive element of the extension  $\mathbb{F}(X_1) \hookrightarrow \mathbb{F}(V)$ , the factors  $m_i$  are pairwise coprime with  $\sum_{i=1}^t \deg_T m_i = \sum_{i=1}^t D_i = D$ .

Consider a single factor, say  $m_1$ , of  $m_{\lambda}$ . A representation of the zeros of  $m_1$  is provided by the well-known Hensel lemma (see, e.g., Abhyankar [1, Lecture 12]). More precisely, assume that

$$(4.10) \quad \text{Res}_T \left( m_1(0, T), \frac{\partial m_1}{\partial T}(0, T) \right) \neq 0.$$

Then there is a factorization  $m_1 = \prod_{i=1}^{D_1} (T - \sigma_i)$  in  $\overline{\mathbb{F}}[[X_1]][T]$ , where  $\overline{\mathbb{F}}[[X_1]]$  denotes the ring of formal power series in  $X_1$  with coefficients in  $\overline{\mathbb{F}}$ , and  $\sigma_i \neq \sigma_j$  for  $i \neq j$ . We shall see that an approximation of order  $2D\delta$  of a single power series, say  $\sigma = \sigma_1$ , suffices to compute  $m_1$ .

For this purpose, we observe that  $m_1 \in \mathbb{F}[X_1, T]$  can also be characterized as the primitive polynomial  $p \in \mathbb{F}[X_1][T]$  of minimal degree (up to nonzero multiples in  $\mathbb{F}$ ) for which  $p(X_1, \sigma) = 0$  holds. Let  $N = 2D\delta$  and let  $\sigma_N \in \mathbb{F}[X_1]$  be the power series  $\sigma$  truncated up to order  $N + 1$ , that is,  $\sigma_N$  is the polynomial of degree at most  $N$  congruent to  $\sigma$  modulo  $X_1^{N+1}$ . Our next result shows that  $m_1$  can be obtained as the solution of a suitable congruence equation involving  $\sigma_N$ .

**Lemma 4.3.** *Let  $p \in \mathbb{F}[X_1, T]$  be a polynomial with  $\deg_{X_1} p \leq \delta$  and  $\deg_T p \leq D$  satisfying the condition*

$$(4.11) \quad p(X_1, \sigma_N) \equiv 0 \pmod{X_1^{N+1}}.$$

*Then  $m_1$  divides  $p$  in  $\mathbb{F}[X_1, T]$ .*

*Proof.* Let  $p \in \mathbb{F}[X_1, T]$  be a solution of (4.11) satisfying the conditions of the lemma. The resultant  $q \in \mathbb{F}[X_1]$  of  $p$  and  $m_1$  with respect to  $T$  has degree at most  $N$  and belongs to the ideal generated by  $m_1$  and  $p$ . Since  $m_1(X_1, \sigma_N) \equiv 0 \pmod{X_1^{N+1}}$  and  $p(X_1, \sigma_N) \equiv 0 \pmod{X_1^{N+1}}$  by hypothesis, we deduce that  $q(X_1) \equiv 0 \pmod{X_1^{N+1}}$ . Therefore, the facts that  $\deg q \leq N$  and  $q(X_1) \equiv 0 \pmod{X_1^{N+1}}$  imply  $q = 0$ . In particular,  $m_1$  and  $p$  have a nonconstant common factor in  $\mathbb{F}(X_1)[T]$ .

Taking into account the irreducibility of  $m_1$  in  $\mathbb{F}(X_1)[T]$  and the Gauss lemma, we easily deduce the statement of the lemma.  $\square$

Lemma 4.3 characterizes  $m_1$  as the nonzero solution of (4.11) of minimal degree in  $T$ , up to nonzero multiples in  $\mathbb{F}$ . We remark that the argument of Lemma 4.3 holds *mutatis mutandis* with any  $M = 2D'\delta'$  instead of  $N$ , where  $D'$  and  $\delta'$  are upper bounds of  $D$  and  $\delta$  respectively.

## 5. THE HOMOTOPY CURVE

This section studies the set  $V^* \subset \overline{\mathbb{F}}^{d+e}$  of common solutions of the equations (4.2) and provides a homotopy between the solutions for  $(\boldsymbol{\alpha}, \boldsymbol{\beta})$  and for  $(\boldsymbol{\alpha}, \mathbf{0})$ . We use throughout the notation given there and assume that  $\boldsymbol{\alpha}$  is generic.

We consider the projection  $\pi_S: V^* \rightarrow \overline{\mathbb{F}}^1$  with  $\pi_S(\mathbf{g}, \mathbf{h}, s) = s$ .

**Lemma 5.1.**  *$\pi_S$  is dominant.*

*Proof.* For  $s \in \overline{\mathbb{F}}$ , we write  $\pi_S^{-1}(s) = W_s \times \{s\}$ , where  $W_s \subset \overline{\mathbb{F}}^{d+e-1}$  is the set of solutions  $(g, h)$  of the system  $(G \circ H)(\alpha_i) - s\beta_i = 0$  for  $1 \leq i \leq d+e-1$ , namely the set of interpolants for the interpolation problem determined by  $(\boldsymbol{\alpha}, s\boldsymbol{\beta})$ . By Corollary 3.4, any  $(\boldsymbol{\gamma}, \boldsymbol{\delta}) \in \mathcal{U}$  admits a nonempty finite set of interpolants. As  $(\boldsymbol{\alpha}, \boldsymbol{\beta})$  belongs to this dense Zariski open subset  $\mathcal{U} \subset \overline{\mathbb{F}}^{2(d+e-1)}$ , it follows that the affine line  $L_{(\boldsymbol{\alpha}, \boldsymbol{\beta})} = \{(\boldsymbol{\alpha}, s\boldsymbol{\beta}): s \in \overline{\mathbb{F}}^1\}$  has a nonempty intersection with  $\mathcal{U}$ . In particular,  $L_{(\boldsymbol{\alpha}, \boldsymbol{\beta})} \cap \mathcal{U}$  is a nonempty Zariski open subset of  $L_{(\boldsymbol{\alpha}, \boldsymbol{\beta})}$ , which proves that there is a nonempty Zariski open subset  $U_S = \{s: (\boldsymbol{\alpha}, s\boldsymbol{\beta}) \in L_{(\boldsymbol{\alpha}, \boldsymbol{\beta})} \cap \mathcal{U}\}$  of  $\overline{\mathbb{F}}^1$ , so that the interpolation problem determined by  $(\boldsymbol{\alpha}, s\boldsymbol{\beta})$  for  $s \in U_S$  admits a finite positive number of interpolants. In particular,  $\pi_S^{-1}(s)$  is nonempty for any  $s \in U_S$ , which proves that  $\pi_S$  is dominant.  $\square$

Let  $V^* = V_1 \cup \dots \cup V_u$  be the decomposition of  $V^*$  into irreducible components defined over  $\mathbb{F}$ . Since  $\pi_S$  is dominant, we may assume that the restriction  $\pi_S|_{V_i}: V_i \rightarrow \overline{\mathbb{F}}^1$  of  $\pi_S$  to  $V_i$  is dominant for  $1 \leq i \leq t$  and not dominant for  $t+1 \leq i \leq u$ , with some  $t \in \{1, \dots, u\}$ . We now define the curve which describes our deformation:

$$(5.1) \quad V = V_1 \cup \dots \cup V_t \subset \overline{\mathbb{F}}^{d+e}.$$

**Lemma 5.2.** *For  $1 \leq i \leq t$ ,  $\dim V_i = 1$ .*

*Proof.* Fix  $i$  with  $1 \leq i \leq t$ . The Theorem on the dimension of fibers (see, e.g., [30, §6.3, Theorem 7]) shows that

$$\dim V_i - 1 = \dim(\pi_S|_{V_i})^{-1}(s)$$

for  $s \in U_S$ . The proof of Lemma 5.1 shows that  $\pi_S^{-1}(s)$  has dimension 0 for generic  $s \in \overline{\mathbb{F}}^1$ , which implies that

$$\dim V_i - 1 = \dim(\pi_S|_{V_i})^{-1}(s) \leq 0.$$

Taking into account that  $V \subset \overline{\mathbb{F}}^{d+e}$  is defined by the  $d+e-1$  equations (4.2), it follows that  $\dim V_i \geq 1$  for  $1 \leq i \leq u$ . We conclude that  $\dim V_i = 1$ .  $\square$

Next we give a more intrinsic definition of the curve  $V$ . Let  $Q_i = (G \circ H)(\alpha_i) - \beta_i S$  for  $1 \leq i \leq d+e-1$ , denote  $\mathbf{Q} = (Q_1, \dots, Q_{d+e-1})$  and let  $J$  be the Jacobian determinant of  $\mathbf{Q}$  with respect to  $\mathbf{G}$  and  $\mathbf{H}$ , that is,

$$J = \det(\partial \mathbf{Q} / \partial (\mathbf{G}, \mathbf{H})).$$

Let  $\mathcal{I} \subset \mathbb{F}[\mathbf{G}, \mathbf{H}, S]$  be the ideal generated by  $Q_1, \dots, Q_{d+e-1}$  and let

$$\mathcal{I} : J^\infty = \{Q \in \mathbb{F}[\mathbf{G}, \mathbf{H}, S] : \exists m \geq 0 \text{ with } J^m Q \in \mathcal{I}\}$$

be the saturation of  $\mathcal{I}$  with respect to  $J$ . Its variety  $\mathcal{Z}(\mathcal{I} : J^\infty) \subset \mathbb{F}^{d+e}$  is the Zariski closure of the locally closed set  $\mathcal{Z}(\mathcal{I}) \setminus \mathcal{Z}(J)$ .

**Lemma 5.3.**  $V = \mathcal{Z}(\mathcal{I} : J^\infty)$ .

*Proof.* For a component  $V_i$  with  $i > t$  we have  $\dim \pi_S(V_i) = 0$ . Further, as  $V_i$  is irreducible, also  $\pi_S(V_i)$  is irreducible, so that  $\pi_S(V_i) = \{s_i\}$  for a specific  $s_i \in \overline{\mathbb{F}}^1$ . The fiber  $\pi_S^{-1}(s_i)$  is the set of common zeros in  $\overline{\mathbb{F}}^{d+e-1}$  of the polynomials  $Q_j(\mathbf{G}, \mathbf{H}, s_i)$  ( $1 \leq j \leq d+e-1$ ). For a given point  $(\mathbf{g}, \mathbf{h}, s_i) \in \pi_S^{-1}(s_i) \cap V_i$ , if  $J(\mathbf{g}, \mathbf{h}, s_i)$  was nonzero, then  $(\mathbf{g}, \mathbf{h}, s_i)$  would be an isolated point of  $V_i$ , contradicting the fact that  $\dim V_i \geq 1$ . It follows that  $J(\mathbf{g}, \mathbf{h}, s_i) = 0$  for any  $(\mathbf{g}, \mathbf{h}, s_i) \in V_i$  and  $\mathcal{Z}(\mathcal{I} : J^\infty) \subseteq V$ .

On the other hand, let  $1 \leq i \leq t$ . By Lemma 5.2,  $V_i$  has dimension 1 and  $\pi_S|_{V_i} : V_i \rightarrow \overline{\mathbb{F}}^1$  is dominant. Since  $V_{t+1} \cup \dots \cup V_u$  projects onto the finite set  $\{s_{t+1}, \dots, s_u\}$ , for  $s \in \overline{\mathbb{F}}^1 \setminus \{s_{t+1}, \dots, s_u\}$  the fiber  $\pi_S^{-1}(s)$  does not intersect  $V_{t+1} \cup \dots \cup V_u$ . Further, since  $\pi_S^{-1}(1) = \pi^{-1}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \times \{1\}$  and  $\#\pi^{-1}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \deg \pi$  by Fact 3.8 and the genericity of  $\boldsymbol{\alpha} \in \mathcal{U}$ , we conclude that  $\#\pi_S^{-1}(s) = \deg \pi$  for generic  $s \in \overline{\mathbb{F}}^1$ . In particular, by Lemma 3.10 we have that  $\pi$  is unramified at any  $(\mathbf{g}, \mathbf{h}, \boldsymbol{\gamma}, \boldsymbol{\delta}) \in \pi^{-1}(\boldsymbol{\gamma}, \boldsymbol{\delta})$  with  $(\boldsymbol{\gamma}, \boldsymbol{\delta}) \in \mathcal{U}$ , which implies that  $\pi_S$  is unramified at any  $(\mathbf{g}, \mathbf{h}, s) \in \pi_S^{-1}(s)$  with  $s$  in the nonempty Zariski open subset  $U_S$  of  $\overline{\mathbb{F}}^1$ .

We pick some  $s \in \pi_S(V_i) \cap U_S$  and  $(\mathbf{g}, \mathbf{h}, s) \in \pi_S^{-1}(s)$ . Then  $J(\mathbf{g}, \mathbf{h}, s) \neq 0$ , so that  $V_i$  is not contained in  $\{J = 0\}$ . It follows that the Zariski closure  $V_i$  of  $V_i \setminus \{J = 0\}$  is contained in the zero set  $\mathcal{Z}(\mathcal{I} : J^\infty)$ , which completes the proof of the lemma.  $\square$

With a slight abuse of notation, we also denote the restriction of  $\pi_S$  to  $V$  as  $\pi_S$ , that is, we consider the projection

$$\pi_S : V \rightarrow \overline{\mathbb{F}}^1 \text{ with } \pi_S(\mathbf{g}, \mathbf{h}, s) = s.$$

As  $V = V_1 \cup \dots \cup V_t$  is the irreducible decomposition of  $V$  over  $\mathbb{F}$  and  $\pi_S|_{V_i} : V_i \rightarrow \overline{\mathbb{F}}^1$  is dominant for  $1 \leq i \leq t$ , we define

$$\deg \pi_S = \sum_{1 \leq i \leq t} \deg \pi_S|_{V_i}.$$

In the next corollary we collect several consequences of Lemma 5.3.

**Corollary 5.4.** *The following assertions hold:*

- (1) *the extension  $\mathbb{F}(S) \hookrightarrow \mathbb{F}(V_i)$  is separable for  $1 \leq i \leq t$ ;*
- (2)  *$\deg \pi_S = \deg \pi$ ;*
- (3)  *$\mathcal{I} : J^\infty$  is a radical ideal.*

*Proof.* Fix  $i$  with  $1 \leq i \leq t$ . Since  $V_i$  is not contained in  $\{J = 0\}$ ,  $V_i \cap \{J = 0\}$  has dimension zero, and thus  $V_i \setminus \{J = 0\}$  is a nonempty Zariski open subset of  $V_i$ . Furthermore,  $\pi_S|_{V_i}$  is unramified precisely at all points of this open subset. According to [21, Proposition T.8], the finite field extension  $\mathbb{F}(S) \hookrightarrow \mathbb{F}(V_i)$  is separable, which proves (1).

As a consequence, for  $1 \leq i \leq t$  there exists a nonempty Zariski open subset  $U_i$  of  $\overline{\mathbb{F}}^1$  such that  $\#(\pi_S|_{V_i})^{-1}(s) = \deg(\pi_S|_{V_i})$  for any  $s \in U_i$ . On the other hand,



$\#\pi^{-1}(s) = \deg \pi$  for any  $s$  in a nonempty Zariski open subset  $U_S \subset \overline{\mathbb{F}^1}$ . Let  $s^* \in U_1 \cap \cdots \cap U_t \cap U_S \setminus \{s_{t+1}, \dots, s_u\}$  be such that  $\pi_S|_{V_i}^{-1}(s^*) \cap \pi_S|_{V_j}^{-1}(s^*) = \emptyset$  for  $i \neq j$ . We have

$$\deg \pi_S = \sum_{i=1}^t \deg \pi_S|_{V_i} = \sum_{i=1}^t \#(\pi_S|_{V_i})^{-1}(s^*) = \#\pi_S^{-1}(s^*) = \#\pi^{-1}(s) = \deg \pi.$$

Finally, since  $\{J = 0\}$  intersects properly each irreducible component  $V_i$  with  $1 \leq i \leq t$ , by [16, Lemma 2.1] we conclude that  $\mathcal{I} : J^\infty$  is a radical ideal.  $\square$

Corollary 5.4 and Lemma 3.5 imply the bounds

$$(5.2) \quad D = \deg \pi_S = \deg \pi \leq \delta = \deg V \leq d^{d+e-1}.$$

Now assume that  $\lambda \in \mathbb{F}^{d+e-1}$  is such that the linear form  $\mathcal{L} = \lambda \cdot (\mathbf{G}, \mathbf{H}) \in \mathbb{F}[\mathbf{G}, \mathbf{H}]$  induces a coordinate function  $\ell$  of  $\mathbb{F}[V]$  which is a primitive element of the extension  $\mathbb{F}(S) \hookrightarrow \mathbb{F}(V_i)$  for  $1 \leq i \leq t$ . In particular, the minimal polynomial  $m_\lambda \in \mathbb{F}[S, T]$  of  $\ell$  satisfies  $\deg_T m_\lambda = D$ .

For any  $s \in \overline{\mathbb{F}^1}$ , we identify each point in the fiber  $\pi_S^{-1}(s)$  with the vector of its  $(\mathbf{g}, \mathbf{h})$ -coordinates and denote by  $W_s \subset \overline{\mathbb{F}^{d+e-1}}$  the set of such points, so that  $\pi_S^{-1}(s) = W_s \times \{s\}$ . In particular,

$$\pi_S^{-1}(0) = W_0 \times \{0\} = \pi^{-1}(\boldsymbol{\alpha}, \mathbf{0}) \times \{0\}, \quad \pi_S^{-1}(1) = W_1 \times \{1\} = \pi^{-1}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \times \{1\}.$$

Let  $\mathbf{\Lambda} = (\Lambda_1, \dots, \Lambda_{d+e-1})$  be a vector of new indeterminates over  $\mathbb{F}$ .

**Lemma 5.5.** *There exists a nonzero polynomial  $R^{(4)} \in \mathbb{F}[\mathbf{\Lambda}]$  of degree at most  $4D^2$  such that for any  $\lambda \in \mathbb{F}^{d+e-1}$  with  $R^{(4)}(\lambda) \neq 0$ , the linear form  $\mathcal{L} = \lambda \cdot (\mathbf{G}, \mathbf{H})$  separates the points of  $W_0 \cup W_1$ .*

*Proof.* We consider the generic linear form  $\mathcal{L}_\Lambda = \mathbf{\Lambda} \cdot (\mathbf{G}, \mathbf{H})$  and define

$$R^{(4)} = \prod_{p \neq q \in W_0 \cup W_1} (\mathcal{L}_\Lambda(p) - \mathcal{L}_\Lambda(q)) \in \mathbb{F}[\mathbf{\Lambda}].$$

For  $p \neq q$ , we have  $\mathcal{L}_\Lambda(p) \neq \mathcal{L}_\Lambda(q)$ . Since  $\#W_0 \cup W_1 \leq 2D$  by Lemma 3.10, it follows that the nonzero polynomial  $R^{(4)} \in \mathbb{F}[\mathbf{\Lambda}]$  has degree at most  $4D^2$ . Any  $\lambda \in \mathbb{F}^{d+e-1}$  not annihilating  $R^{(4)}$  defines a linear form  $\mathcal{L}_\lambda$  that separates the points of  $W_0 \cup W_1$ .  $\square$

## 6. AN ALGORITHM FOR DECOMPOSABLE INTERPOLATION

The algorithm for computing an interpolant  $(g, h)$  for the input  $(\boldsymbol{\alpha}, \boldsymbol{\beta})$  with a generic  $\boldsymbol{\alpha}$  proceeds as follows. Computing in  $\mathbb{F}[\mathbf{G}, \mathbf{H}, S]$  is straightforward, and we can also take its elements modulo the ideal  $\mathcal{I}(V)$  of the homotopy curve  $V$  to  $\mathbb{F}[V] = \mathbb{F}[\mathbf{G}, \mathbf{H}, S]/\mathcal{I}(V)$ . But testing whether an element of  $\mathbb{F}[\mathbf{G}, \mathbf{H}, S]$  is zero in  $\mathbb{F}[V]$  amounts to an ideal membership test, which is a nontrivial task. In order to circumvent this, we work with power series expansions. Recall the general assumption that  $\text{char}(\mathbb{F})$  does not divide  $d$ .

From Section 2, we find an interpolating polynomial  $(g_0, h_0) \in W_0$  for  $(\boldsymbol{\alpha}, \mathbf{0})$ , so that  $\mathbf{v} = (g_0, h_0, 0) \in V$ . Now  $\mathbf{v}$  actually lies on a single irreducible component of  $V$ , say  $V_1$ . Furthermore,  $S \in \mathbb{F}[\mathbf{G}, \mathbf{H}, S]$  is a local parameter on  $V_1$  at  $\mathbf{v}$ , so that there is an injective morphism of  $\mathbb{F}$ -algebras  $\tau : \mathcal{O}_{\mathbf{v}} \rightarrow \mathbb{F}[[Y]]$  from the local ring  $\mathcal{O}_{\mathbf{v}}$  of  $V_1$  at  $\mathbf{v}$  to the ring of power series in a new variable  $Y$ . Since  $\tau(S) = Y$  generates this ring,  $\tau$  is actually an isomorphism.

To simplify notation, we abbreviate  $\mathbf{G}|\mathbf{H} = (G_{d-1}, \dots, G_0, H_{e-1}, \dots, H_1)$ , a vector of  $d + e - 1$  indeterminates. For each  $i < d$ ,  $\gamma_i = \tau(G_i) \in \mathbb{F}[[Y]]$  is a power series, and similarly  $\eta_i = \tau(H_i) \in \mathbb{F}[[Y]]$  for  $i < e$ . We abbreviate again  $\boldsymbol{\gamma}|\boldsymbol{\eta} = (\gamma_{d-1}, \dots, \gamma_0, \eta_{e-1}, \dots, \eta_1)$ .

A Newton-Hensel iteration computes an approximation of  $\boldsymbol{\gamma}|\boldsymbol{\eta}$  to a sufficiently large precision  $N$ , that is,  $\boldsymbol{\psi} \in (\mathbb{F}[Y]/(Y^N))^{d+e-1}$  with  $\boldsymbol{\psi} \equiv \boldsymbol{\gamma}|\boldsymbol{\eta} \pmod{Y^N}$ . For a randomly chosen  $\boldsymbol{\lambda} \in \mathbb{F}^{d+e-1}$ , we take the linear combination  $\boldsymbol{\lambda} \cdot (\boldsymbol{\gamma}|\boldsymbol{\eta}) = \sum_{1 \leq i \leq d} \lambda_i \gamma_{d-i} + \sum_{d < i < d+e} \lambda_i \eta_{d+e-i}$ , its truncation  $L = \sum_{1 \leq i < d+e} \lambda_i \psi_i \in \mathbb{F}[Y]/(Y^N)$  and find the minimal polynomial  $m \in \mathbb{F}[Y, T]$  of  $\boldsymbol{\lambda} \cdot \boldsymbol{\gamma}|\boldsymbol{\eta}$ , which equals the polynomial of minimal degree in  $T$  with  $\deg m \leq d^{d+e-1}$  such that  $m(Y, L) \equiv 0 \pmod{Y^N}$ . Furthermore, we calculate  $m'((\mathbf{G}|\mathbf{H})_i)$  for each  $i < e + d$ , where  $m' = \partial m / \partial T$  is the derivative of  $m$ . These data provide a geometric solution of  $V_1$ , and it then remains to substitute  $S = 1$  in them to find the desired interpolation polynomial  $(g, h)$  for  $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ .

We sketch the whole algorithm as follows.

**Algorithm 6.1.**

Input: Generic  $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathbb{F}^{d+e-1} \times \mathbb{F}^{d+e-1}$ , real  $\epsilon$  with  $0 < \epsilon < 1$ , access to a finite set  $\mathcal{S} \subseteq \mathbb{F}$  with at least  $32\epsilon^{-1}(d^2 + de)^2 d^{5(d+e-1)}$  elements.

Output: A composite interpolating polynomial  $f$  for  $(\boldsymbol{\alpha}, \boldsymbol{\beta})$  as in Problem 1.1, or “fail”.

- (1) Compute a point  $(\mathbf{g}_0, \mathbf{h}_0) \in W_0$ .
- (2) Set  $M = d^{d+e-1} + 1$  and  $N = 2d^{2(d+e-1)} + 1$ . Compute truncated power series approximations  $\boldsymbol{\gamma}|\boldsymbol{\eta} \in (\mathbb{F}[Y]/(Y^N))^{d+e-1}$  for  $\mathbf{G}|\mathbf{H}$ , with  $\lceil \log_2 N \rceil$  steps of the following quadratic Newton-Hensel iteration for solving the equations  $\mathbf{P} = (P_1, \dots, P_{d+e-1}) = 0$ , where  $P_j(\mathbf{G}, \mathbf{H}) = (G \circ H)(\alpha_j) - \beta_j S \in \mathbb{F}[G, H, S]$  for  $1 \leq j \leq d + e - 1$ . The initial value is  $(\mathbf{g}_0, \mathbf{h}_0)$ . One iteration step is given by the Newton-Hensel operator

$$N_{\mathbf{P}}(\boldsymbol{\gamma}, \boldsymbol{\eta}) = (\boldsymbol{\gamma}, \boldsymbol{\eta}) - (J_{\mathbf{P}}^{-1}(\boldsymbol{\gamma}, \boldsymbol{\eta}, Y)) \left( \mathbf{P}(\boldsymbol{\gamma}, \boldsymbol{\eta}, Y)^{\top} \right) \in \mathbb{F}[Y]^{d+e-1},$$

where  $J_{\mathbf{P}} = \partial \mathbf{P} / \partial (G|H) \in \mathbb{F}[G, H, S]^{(d+e-1) \times (d+e-1)}$  is the Jacobian matrix of the system and  $\top$  denotes transposition. Let  $\boldsymbol{\psi} \in (\mathbb{F}[Y]/(Y^N))^{d+e-1}$  be the resulting vector of power series truncated up to order  $N$ .

- (3) Choose  $\boldsymbol{\lambda} \in \mathcal{S}^{d+e-1}$  uniformly at random. For  $1 \leq j < d + e$ , let

$$\mathbf{Z}^{(j)} = (\lambda_1, \dots, \lambda_{j-1}, \lambda_j + \Lambda_j, \lambda_{j+1}, \dots, \lambda_{d+e-1}),$$

where  $\Lambda_1, \dots, \Lambda_{d+e-1}$  are new indeterminates. Compute

$$\begin{aligned} L^{(j)} &= \mathbf{Z}^{(j)} \cdot \boldsymbol{\psi} = \sum_{1 \leq i < d+e} \mathbf{Z}_i^{(j)} \psi_i \in \mathbb{F}[\Lambda_j][Y]/(Y^N), \\ (L^{(j)})^2, \dots, (L^{(j)})^{2M} &\in \mathbb{F}[\Lambda_j][Y]/(Y^N, \Lambda_j^2). \end{aligned}$$

- (4) For each  $j \in \{1, \dots, d + e - 1\}$ , find by binary search the smallest value  $k_j$  of  $\ell$  for which

$$(6.1) \quad \sum_{0 \leq i \leq \ell} a_i^{(j)} (L^{(j)})^i \equiv 0 \pmod{(Y^N, \Lambda_j^2)}$$

has a solution  $a_0^{(j)}, \dots, a_{\ell}^{(j)} \in \mathbb{F}[\Lambda_j][Y]$  with  $\deg_Y a_i^{(j)} < M$  for  $i \leq \ell$  and  $a_{\ell}^{(j)}$  monic. This corresponds to a system of linear equations for the coefficients

in  $\mathbb{F}[\Lambda_j]/(\Lambda_j^2)$  of the  $a_i^{(j)}$ 's. Compute such a solution  $a_0^{(j)}, \dots, a_{k_j}^{(j)}$  with  $a_{k_j}^{(j)}$  monic of minimal degree  $M_j$ . [Then  $\ell \leq 2M$  and  $M_j \leq 2M$  for all  $\ell$  and  $M_j$  considered.]

(5) For each  $j \in \{1, \dots, d+e-1\}$ , let

$$m^{(j)} = \sum_{0 \leq i \leq k_j} a_i^{(j)} T^i \in \mathbb{F}[\Lambda_j][Y, T], \quad w^{(j)} = -\partial m^{(j)} / \partial \Lambda_j \in \mathbb{F}[\Lambda_j][Y, T].$$

Set

$$m = m^{(1)}|_{\Lambda_1 \leftarrow 0} \in \mathbb{F}[Y, T], \quad v_j = w^{(j)}|_{\Lambda_j \leftarrow 0} + Y \cdot (\partial m / \partial T) \in \mathbb{F}[Y, T].$$

[Then  $\lambda, m, v_1, \dots, v_{d+e-1}$  form a geometric solution of  $V_1$ .]

(6) Substitute 1 for  $Y$  in the polynomials  $m, v_1, \dots, v_{d+e-1}$ . Let  $m_1 = m(1, T)$ . Compute polynomials  $u, w_1, \dots, w_{d+e-1} \in \mathbb{F}[T]$  of degrees less than  $\deg m_1$  that satisfy

$$u \equiv (m_1')^{-1} \pmod{m_1}, \quad w_j \equiv (m_1')^{-1} \cdot v_j(1, T) \pmod{m_1} \quad (1 \leq j < d+e).$$

[Then  $\lambda, m_1, w_1, \dots, w_{d+e-1} \in \mathbb{F}[T]$  form a geometric solution of the zero-dimensional variety  $(W_1 \times \{1\}) \cap V_1$ .]

(7) Find a root  $\eta$  of  $m_1$  in some extension field  $\mathbb{K}$  of  $\mathbb{F}$  and compute

$$g|h = (w_1(\eta), \dots, w_{d+e-1}(\eta)) \in \mathbb{K}^{d+e-1}.$$

If  $(g \circ h)(\alpha_j) = \beta_j$  for  $1 \leq j < d+e$ , then return  $(g, h)$  else return “fail”.

Details on performing some of the steps are given below. We start with an illustration of the first steps of the algorithm.

**Example 6.2.** Continuing Example 2.4, we have  $d = e = 2$ ,  $\alpha = (5, 6, 7)$  and  $\beta = (3, 3, 3)$ . Now we need this for  $\beta = (0, 0, 0)$  and so have to subtract 3 from  $g_0$ . Then the very first solution yields  $(g_0, h_0) = (58, 840, -11)$ . Then for  $\beta = (1, 2, 3)$ , we have

$$J_{\mathcal{P}}(g_0, h_0) = \begin{pmatrix} h_0(\alpha_1) & 1 & g_0'(h_0(\alpha_1))\alpha_1 \\ h_0(\alpha_2) & 1 & g_0'(h_0(\alpha_2))\alpha_2 \\ h_0(\alpha_3) & 1 & g_0'(h_0(\alpha_3))\alpha_3 \end{pmatrix} = \begin{pmatrix} -30 & 1 & -10 \\ -30 & 1 & -12 \\ -28 & 1 & 14 \end{pmatrix},$$

$$J_{\mathcal{P}}^{-1}(g_0, h_0) = \begin{pmatrix} -\frac{13}{2} & 6 & \frac{1}{2} \\ -189 & 175 & 15 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix},$$

$$\mathcal{P}(g_0, h_0) = \begin{pmatrix} g_0 \circ h_0(\alpha_1) - \beta_1 Y \\ g_0 \circ h_0(\alpha_2) - \beta_2 Y \\ g_0 \circ h_0(\alpha_3) - \beta_3 Y \end{pmatrix} = \begin{pmatrix} -\beta_1 Y \\ -\beta_2 Y \\ -\beta_3 Y \end{pmatrix} = \begin{pmatrix} -1 Y \\ -2 Y \\ -3 Y \end{pmatrix},$$

$$\begin{aligned} N_{\mathcal{P}}(g_0, h_0) &= \begin{pmatrix} 58 \\ 840 \\ -11 \end{pmatrix} - \begin{pmatrix} -\frac{13}{2} & 6 & \frac{1}{2} \\ -189 & 175 & 15 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} -1 Y \\ -2 Y \\ -3 Y \end{pmatrix} \\ &= \begin{pmatrix} 58 + 7 Y \\ 840 + 206 Y \\ -11 - \frac{1}{2} Y \end{pmatrix} \equiv \begin{pmatrix} \gamma_1 \\ \gamma_0 \\ \eta_1 \end{pmatrix} \pmod{Y^2}. \end{aligned}$$

Now suppose that the iteration stops here and in the algorithm, we choose  $\lambda = (-1, 1, 2)$ . Then we have

$$(6.2) \quad L = -1 \cdot (58 + 7Y) + 1 \cdot (840 + 206Y) + 2 \cdot \left(-11 - \frac{1}{2}Y\right) = 760 + 198Y \in \mathbb{F}[Y].$$

□

We next provide some subroutines used in Algorithm 6.1 and estimate error probabilities and cost. For the latter, we only count arithmetic operations in  $\mathbb{F}$  and ignore the (small) cost of Boolean operations and finding random elements. Furthermore, we use the dense representation of (multivariate) polynomials and mainly standard procedures for simplicity, except that we denote by  $M(n)$  the cost of multiplying two univariate polynomials of degree at most  $n$ , and by  $\omega$  a feasible exponent for matrix multiplication, so that two  $n \times n$  matrices can be multiplied with  $n^\omega$  operations. Thus we may use  $\omega = 2.3728639$  (Le Gall [12]), and with the notation  $O^\sim(f) = \{g: g \in f(\log f)^{O(1)}\}$  for real functions  $f$  and  $g$ , we have  $M \in O^\sim(n)$ . Possible algorithmic improvements are discussed in Section 7.

For the analysis of the algorithm, we assume throughout that  $\alpha \in \mathbb{F}^{d+e-1}$  is generic, as required.

**Lemma 6.3.** *Steps (1) and (2) can be performed with  $O^\sim(d^{2(d+e-1)})$  operations in  $\mathbb{F}$ .*

*Proof.* In step (1) we use Proposition 2.3 to compute a solution  $(\mathbf{g}_0, \mathbf{h}_0)$  for the interpolation problem given by  $(\alpha, \beta \cdot 0) = (\alpha, \mathbf{0})$ . This requires a  $d$ -refinement of  $\mathcal{P}_0$ , say  $\mathcal{P} = \{\{1\}, \dots, \{d-1\}, \{d, \dots, d+e-1\}\}$ . The system (2.6) of linear equations can be solved for  $\mathbf{h}_0$  with  $(e-1)^\omega$  operations, and the interpolation for  $\mathbf{g}_0$  with  $O(M(d) \log d)$  operations.

One step of the Newton-Hensel iteration in step (2) at most doubles the degree of the  $d+e-1$  power series in  $Y$ . In the first step, given in Example 6.2, this degree is at most 1, and thus in the  $k$ th step at most  $2^k$ . Writing  $q = d+e-1$  for simplicity, the rows of the  $q \times q$  Jacobian matrix  $J_{\mathcal{P}}$  are like the top rows in (3.5). At the start of the  $k$ th step, each coefficient  $(\mathbf{g}|\mathbf{h})_i$  is replaced by a polynomial in  $Y$  of degree at most  $2^{k-1}$ , say by  $(\gamma^{(k-1)}|\boldsymbol{\eta}^{(k-1)})_i$ . The linear combination of the  $\eta_i^{(k-1)}$  corresponding to  $h(\alpha_j)$  with coefficients  $\alpha_j^i$  can be computed with  $O(e2^k)$  operations, and then the powers  $h(\alpha_j)^2, \dots, h(\alpha_j)^{d-1}$  with  $O(deM(2^k))$  further steps. All this needs to be done for the  $q$  values of  $j$ , at a cost of  $O(deqM(2^k))$  operations. Similarly, a single  $g'(h(\alpha_j))$ , given the powers of  $h(\alpha_j)$ , corresponds to at most  $d$  multiplications of polynomials modulo  $Y^{2^k}$  at a cost of  $O(dM(2^k))$  operations. The multiplications by powers of  $\alpha_j$  are even cheaper. This leads to  $O(dqM(2^k))$  operations.

Thus the matrix  $J_{\mathcal{P}}(\gamma^{(k-1)}, \boldsymbol{\eta}^{(k-1)}, Y) \in \mathbb{F}[Y]^{q \times q}$  can be calculated with  $O(deM(2^k))$  operations. It is invertible modulo  $Y$ , hence also modulo  $Y^{2^k}$ . Its inverse in  $\mathbb{F}[Y]^{q \times q}$  can be found with  $O(q^\omega M(2^k))$  operations in  $\mathbb{F}$  and the product by  $\mathbf{P}(\gamma^{(k-1)}, \boldsymbol{\eta}^{(k-1)}, S)^\top$  with  $q^2M(2^k)$  operations.

In total, the  $k$ th step of the iteration takes  $q^\omega M(2^k)$  operations. Since  $M(n)$  is essentially linear,  $\sum_{1 \leq k \leq \kappa} M(2^k) \in O(M(N))$  and the total cost for step (2) is  $O(q^\omega M(N))$ . □

Now  $\pi_S$  is unramified at  $(\mathbf{g}_0, \mathbf{h}_0, 0) \in V = V_1 \cup \dots \cup V_t$  if and only if  $\pi$  is unramified at  $(\mathbf{g}_0, \mathbf{h}_0, \alpha, 0 \cdot \beta) = (\mathbf{g}_0, \mathbf{h}_0, \alpha, \mathbf{0})$ , and the latter holds by Lemma 3.6.

Hence  $(\mathbf{g}_0, \mathbf{h}_0, 0)$  is contained in exactly one of the components. Their numbering is arbitrary and we may assume that  $(\mathbf{g}_0, \mathbf{h}_0, 0) \in V_1$ .

In order to provide some details for step (4), we first describe an algorithm that works for indeterminates  $\mathbf{\Lambda}$  instead of values  $\boldsymbol{\lambda} \in \mathbb{F}^{d+e-1}$ . Its intermediate results may become too large and it is not suited for implementation, but specializations of it are. So we consider a polynomial ring  $R = \mathbb{F}[\mathbf{\Lambda}]$  in variables  $\mathbf{\Lambda} = (\Lambda_1, \dots, \Lambda_{d+e-1})$ ,  $\boldsymbol{\psi} \in (\mathbb{F}[Y]/(Y^N))^{d+e-1}$ , and  $\mathbf{\Lambda} \cdot \boldsymbol{\psi} = \sum_{1 \leq i < d+e} \Lambda_i \psi_i \in \mathbb{F}[\mathbf{\Lambda}][Y]/(Y^N)$ .

**Subroutine 6.4.**

Input:  $\boldsymbol{\psi} \in (\mathbb{F}[Y]/(Y^N))^{d+e-1}$ .

Output: Integer  $k$  and a polynomial  $\mu \in \mathbb{F}(\mathbf{\Lambda})[Y, T]$  with  $\deg_T \mu = k$ .

(S1) Find the smallest value  $k$  of  $\ell$  for which

$$(6.3) \quad \sum_{0 \leq i \leq \ell} A_i(\mathbf{\Lambda} \cdot \boldsymbol{\psi})^i \equiv 0 \pmod{Y^N}$$

has a solution  $A_0, \dots, A_\ell \in \mathbb{F}(\mathbf{\Lambda})[Y]$  with  $\deg_Y A_i < M$  for all  $i \leq \ell$  and  $A_\ell$  monic, and compute such a solution  $A_0, \dots, A_k$  with  $A_k$  monic of minimal degree  $M'$ . Starting with  $\ell = 1, 2, 4, \dots$ , this binary search keeps on doubling the values of  $\ell$  while calls to the subroutine (S2) return “fail”. Then it backtracks and homes in on the smallest values  $k$  and similarly finds the smallest  $M'$  for which (S2) returns an output.

(S2) The equation (6.3) corresponds to a system of  $N$  linear equations for the  $\ell M + M'$  coefficients of all  $A_i \in \mathbb{F}(\mathbf{\Lambda})[Y]$  with  $\deg_Y A_i < M$  for  $i < \ell$  and  $\deg_Y A_\ell < M'$ . With a suitable matrix  $B \in \mathbb{F}(\mathbf{\Lambda})^{N \times (\ell M + M')}$ , the vector  $A \in \mathbb{F}(\mathbf{\Lambda})^{\ell M + M'}$ , and a suitable constant vector  $C \in \mathbb{F}(\mathbf{\Lambda})^N$ , this corresponds to the system  $BA = C$ . Compute with a version of Gaussian elimination matrices  $D \in \mathbb{F}(\mathbf{\Lambda})^{N \times (\ell M + M')}$ ,  $P \in \mathbb{F}(\mathbf{\Lambda})^{(\ell M + M') \times (\ell M + M')}$  and a vector  $E \in \mathbb{F}(\mathbf{\Lambda})^N$  so that  $BX = C$  is equivalent to  $DPX = E$  for all  $X \in \mathbb{F}^{\ell M + M'}$ ,  $P$  is a permutation matrix, and  $D$  and  $E$  have the form:

$$\begin{array}{|c|c|c|} \hline \star & * & * \\ \hline & \ddots & \vdots \\ \hline 0 & \ddots & \vdots \\ \hline & & * \\ \hline 0 & & 0 \\ \hline \end{array} \begin{array}{|c|} \hline * \\ \hline * \\ \hline * \\ \hline * \\ \hline * \\ \hline \end{array},$$

where  $*$  stands for an arbitrary value and  $\star$  for a nonzero value in  $\mathbb{F}(\mathbf{\Lambda})$ . If the bottom entries of  $E$  (designated by a single  $*$ ) are all zero, then compute by back-substitution a solution  $Y \in \mathbb{F}(\mathbf{\Lambda})^{\ell M + M'}$  of  $DY = E$  with the value 0 for those entries of  $Y$  that correspond to the lower block of the matrix,  $A = P^{-1}Y \in \mathbb{F}(\mathbf{\Lambda})^{\ell M + M'}$  and return  $A$ , else return “fail”.

(S3) Return  $k$  and  $\mu = \sum_{0 \leq i \leq k} A_i T^i \in \mathbb{F}(\mathbf{\Lambda})[Y, T]$ .

The Gaussian elimination works row by row, choosing a suitable nonzero pivot element  $p \in \mathbb{F}(\mathbf{\Lambda})$ , multiplying one row by the pivot and then adding it to another row. This creates entries with value 0 in the matrix.

Now step (4) of Algorithm 6.1 is implemented by running Subroutine 6.4 on the special inputs  $\mathbf{Z}^{(j)}$ . Gauss elimination performs two types of computations:

calculations in the ground field, here  $\mathbb{F}(\mathbf{\Lambda})$ , and tests for zero. Such algorithms are sometimes called *arithmetic-Boolean circuits*. If  $\mathbf{\Lambda}$  is substituted by some special value, say  $\boldsymbol{\lambda}$  or  $\mathbf{Z}^{(j)}$ , then some of the tests that originally return “ $\neq 0$ ” might return “ $= 0$ ”. In (6.4) below we define a polynomial  $R^{(5)} \in \mathbb{F}[\mathbf{\Lambda}]$  such that  $R^{(5)}(\boldsymbol{\lambda}) \neq 0$  guarantees that this does not happen. We call some  $\boldsymbol{\lambda} \in \mathbb{F}^{d+e-1}$  *lucky* if  $R^{(5)}(\boldsymbol{\lambda}) \neq 0$ , and show that the latter occurs with high probability.

In other words, the condition  $R^{(5)}(\boldsymbol{\lambda}) \neq 0$  fixes the Boolean sequence of test outcomes and turns the arithmetic-Boolean circuit into an arithmetic circuit (or straight-line program).

**Lemma 6.5.** *For the output of Subroutine 6.4, we have  $k \leq d^{d+e-1}$ .  $R^{(5)} \in \mathbb{F}[\mathbf{\Lambda}]$  is a nonzero polynomial of degree at most  $32(d^2 + de)^2 d^{5(d+e-1)}$ . A  $\boldsymbol{\lambda}$  as chosen in step (3) of Algorithm 6.1 is lucky with probability at least  $1 - \epsilon$ . For any lucky  $\boldsymbol{\lambda}$ , the following statements hold:*

- (1) *The linear form  $\mathcal{L} = \boldsymbol{\lambda} \cdot (\mathbf{G}, \mathbf{H}) \in \mathbb{F}[\mathbf{G}, \mathbf{H}]$  separates the points of  $W_0 \cup W_1$ .*
- (2) *For  $1 \leq j < d + e$ , we have  $k_j = k$ ,  $m^{(j)} = \mu|_{\mathbf{\Lambda} \leftarrow \mathbf{Z}^{(j)}}$  and  $m = \mu|_{\mathbf{\Lambda} \leftarrow \boldsymbol{\lambda}}$*

*Proof.* From [29, Proposition 1] we know that there exists a solution to (6.1) with  $k \leq D \leq d^{d+e-1}$  and  $M' \leq \delta \leq d^{d+e-1}$ . Therefore  $\max\{k, M'\} \leq d^{d+e-1} < M$  in (6.3), so that  $\max\{\ell, M'\}$  never exceeds  $2M$ .

For the probabilistic analysis in (1), Lemma 5.5 provides a nonzero polynomial  $R^{(4)} \in \mathbb{F}[\mathbf{\Lambda}]$  of total degree at most  $4D^2 \leq 4d^{2(d+e-1)}$  so that  $R^{(4)}(\boldsymbol{\lambda}) \neq 0$  is sufficient for the linear form  $\mathcal{L} = \boldsymbol{\lambda} \cdot (\mathbf{G}, \mathbf{H})$  to separate the points of  $W_0 \cup W_1$ .

We now present a similar polynomial  $R^{(5)} \in \mathbb{F}[\mathbf{\Lambda}]$  for the claim in (2), namely, such that for any  $\boldsymbol{\lambda} \in \mathbb{F}^{d+e-1}$  with  $R^{(5)}(\boldsymbol{\lambda}) \neq 0$ , all pivots and denominators in the subroutine with  $\boldsymbol{\lambda}$  substituted for  $\mathbf{\Lambda}$  are nonzero. A fortiori, this holds for substitution by any  $\mathbf{Z}^{(j)}$ .

We represent each intermediate result in  $\mathbb{F}(\mathbf{\Lambda})$  as the quotient of two coprime polynomials in  $\mathbb{F}[\mathbf{\Lambda}]$ . The *degree* of the rational function is the maximum of the two polynomial degrees. Each calculation step yields first a quotient of two arbitrary polynomials which we then simplify by dividing out their gcd. All pivots and divisors in the Gaussian elimination of step (S2) for some value of  $\ell$  are quotients of determinants of two square submatrices of the original coefficient matrix  $B$ ; see Edmonds [10] and Bareiss [2]. In (6.3), the (total) degree in  $\mathbf{\Lambda}$  is at most  $\ell$ , so this also holds for all entries in  $B$ . The size of square submatrices is at most  $N \times N$ , and their determinants have degree at most  $\ell N$  in  $\mathbf{\Lambda}$ . We denote by  $R_{\ell, M'}^{(5)} \in \mathbb{F}[\mathbf{\Lambda}]$  the product of all pivots examined and found to be nonzero in the course of the subroutine. Since there are at most  $N$  such pivots, we have  $\deg R_{\ell, M'}^{(5)} \leq \ell N^2 \leq 2d^{d+e-1} N^2$ . At most  $2 \log_2 2d^{d+e-1} \leq 2(d+e) \log_2 d$  values of  $\ell$  and  $M'$  are used. We let

$$(6.4) \quad R^{(5)} = R^{(4)} \cdot \prod_{\ell, M'} R_{\ell, M'}^{(5)},$$

where the product runs over all such  $\ell$ .  $R^{(5)}$  is a nonzero polynomial of degree at most

$$\begin{aligned} 4d^{2(d+e-1)} + 4(\log_2 d)^2 (d+e)^2 \cdot 2d^{d+e-1} N^2 \\ \leq 32(d^2 + de)^2 \cdot d^{5(d+e-1)}. \end{aligned}$$

It is well known (see for example von zur Gathen & Gerhard [14, Lemma 6.44]) that for a nonzero polynomial  $p \in \mathbb{F}[Y_1, \dots, Y_n]$  and a uniformly random choice of  $y = (y_1, \dots, y_n) \in \mathcal{S}^n \subseteq \mathbb{F}^n$ ,

$$\text{prob}\{p(y) = 0\} \leq \text{deg } p / \#\mathcal{S},$$

where  $\text{deg}$  denotes the total degree. This implies the second claim.  $\square$

In particular, in step (5) of Algorithm 6.1, we have  $m = m^{(j)}|_{\Lambda_j \leftarrow 0}$  for any  $j < d + e$ .

**Lemma 6.6.** *Step (3) can be performed with  $O^\sim(d^{3(d+e-1)})$  operations in  $\mathbb{F}$  for  $1 \leq j < d + e$ .*

*Proof.* We perform  $(d + e)M$  multiplications of univariate polynomials in  $\mathbb{F}[\Lambda_j][Y]$ , each with degree at most 1 in  $\Lambda_j$  and at most  $N$  in  $Y$ . A single such multiplication can be done with  $O^\sim(N)$  operations, for a total cost of  $O^\sim(d^{3(d+e-1)})$  operations in  $\mathbb{F}$ .  $\square$

**Lemma 6.7.** *For a lucky choice of  $\lambda$ , step (4) can be performed with  $O^\sim(d^{6(d+e-1)})$  operations in  $\mathbb{F}$ , for  $1 \leq j < d + e$ .*

*Proof.* If (6.1) has a solution for some values  $\ell$  and  $M'$ , then it also has one for  $\ell + 1$  and  $M' + 1$  and all larger values of  $\ell$  and  $M'$ . Therefore, by Lemma 6.5, in the binary search in step (4) all values  $\ell$  and  $M'$  are at most  $2d^{d+e-1}$ .

We first consider (6.1) for an arbitrary value of  $\ell$  and  $M'$  and write each unknown  $a_i^{(j)} \in \mathbb{F}[\Lambda_j][Y]$  as  $a_i^{(j)} = \sum_{0 \leq m < M} a_{im}^{(j)} Y^m$  for  $i < \ell$  and  $a_\ell^{(j)} = \sum_{0 \leq m < M'} a_{\ell m}^{(j)} Y^m$ , and  $(L^{(j)})^i = \sum_{0 \leq h < N} \ell_{ih}^{(j)} Y^h \text{ mod } (Y^N, \Lambda_j^2)$ , with all  $\ell_{ih}^{(j)} \in \mathbb{F}[\Lambda_j]$  polynomials of degree at most 1 and unknowns  $a_{im}^{(j)}$  taking values in  $\mathbb{F}[\Lambda_j]/(\Lambda_j^2)$ . Thus (6.1) corresponds to a system of  $N$  linear equations in  $\ell M + M'$  unknowns over  $\mathbb{F}[\Lambda_j]/(\Lambda_j^2)$ .

We use the calculation of Subroutine 6.4 with input  $\mathbf{Z}^{(j)}$  instead of  $\Lambda$ . By the assumption that the choice of  $\lambda$  is lucky, all rational functions of  $\mathbb{F}(\Lambda_j)$  arising during the execution of Subroutine 6.4 on input  $\mathbf{Z}^{(j)}$  are well-defined modulo  $\Lambda_j$ , which implies that they are well-defined modulo  $\Lambda_j^2$ . We conclude that the output of step (4) is of the form

$$(6.5) \quad m^{(j)}|_{\Lambda_j \leftarrow 0} + (\partial m^{(j)} / \partial \Lambda_j)|_{\Lambda_j \leftarrow 0} \cdot \Lambda_j.$$

The system of linear equations is of size  $(\ell M + M') \times N$  and  $\ell M + M', N \leq 3d^{2(d+e-1)}$ . Solving it can be done with  $27d^{6(d+e-1)}$  calculations on matrix entries, which are elements of the quotient ring  $\mathbb{F}[\Lambda_j]/(\Lambda_j^2)$ . More precisely, each operation involves the images modulo  $\Lambda_j^2$  of two rational functions of  $\mathbb{F}(\Lambda_j)$  (which are well-defined modulo  $\Lambda_j^2$ ), and is performed modulo  $\Lambda_j^2$ . Since each operation in  $\mathbb{F}(\Lambda_j)$  is replaced by  $O(1)$  operations in  $\mathbb{F}$ , we conclude that the whole procedure requires  $O(d^{6(d+e-1)})$  operations in  $\mathbb{F}$ . At most  $2 \log_2(2d^{d+e-1})$  values of  $\ell$  and  $M'$  occur and the claimed bound on the running time follows.  $\square$

**Lemma 6.8.** *For a lucky choice of  $\lambda$ , the values computed in step (5) constitute a geometric solution of  $V_1$ . The computation can be performed with  $O^\sim(d^{2(d+e-1)})$  operations in  $\mathbb{F}$ .*

*Proof.* For  $1 \leq j < d+e$ , the polynomial  $m^{(j)}$  has  $\ell M + M' \leq 3d^{2(d+e-1)}$  coefficients in  $\mathbb{F}[\Lambda_j]/(\Lambda_j^2)$ . By the expression (6.5) of  $m^{(j)}$  the claim on the number of operations in  $\mathbb{F}$  follows. The fact that the output of step (5) is a geometric solution of  $V_1$  is due to (4.8) and (4.9).  $\square$

**Lemma 6.9.** *For a lucky choice of  $\lambda$ , the output of step (6) is a geometric solution of the zero-dimensional variety  $(W_1 \times \{1\}) \cap V_1$ . It can be performed with  $O^\sim(d^{3(d+e-1)})$  operations in  $\mathbb{F}$ .*

*Proof.* Recall that  $(\pi_S|_{V_1})^{-1}(1) = (W_1 \times \{1\}) \cap V_1$ . Observe that  $\#((W_1 \times \{1\}) \cap V_1) = \deg \pi_S|_{V_1} = D_1 > 0$  and all the elements of  $\pi_S((W_1 \times \{1\}) \cap V_1)$  are interpolants for the interpolation problem defined by  $(\alpha, \beta)$ . Since  $\lambda, m, v_1, \dots, v_{d+e-1}$  form a geometric solution of  $V_1$ , for  $m_1 = m(1, T)$  we have

$$(W_1 \times \{1\}) \cap V_1 = \{(1, \mathbf{g}, \mathbf{h}) \in \mathbb{F}^{d+e-1} : m_1(\mathcal{L}(\mathbf{g}, \mathbf{h})) = 0, \\ m'_1(\mathcal{L}(\mathbf{g}, \mathbf{h}))(g|h)_j = v_j(1, \mathcal{L}(\mathbf{g}, \mathbf{h})) \ (1 \leq j < d+e)\}.$$

Observe that  $\deg m_1 \leq D_1$ ,  $m_1 \neq 0$  and  $m_1$  vanishes on the set  $\{\mathcal{L}(\mathbf{g}, \mathbf{h}) : (\mathbf{g}, \mathbf{h}) \in W_1\}$ , of cardinality  $D_1$ . It follows that  $\deg m_1 = D_1$ ,  $m_1$  and  $m'_1$  are coprime, and thus the inverse  $(m'_1)^{-1}$  modulo  $m_1$  is well-defined. In particular, defining  $w_j = (m'_1)^{-1} \cdot v_j$  modulo  $m_1$  for  $1 \leq j < d+e$ , we obtain

$$(W_1 \times \{1\}) \cap V_1 = \{(1, w_1(\eta), \dots, w_{d+e-1}(\eta)) \in \mathbb{F}^{d+e} : \eta \in \overline{\mathbb{F}}, m_1(\eta) = 0\}.$$

This shows that  $\lambda, m_1, w_1, \dots, w_{d+e-1}$  form a geometric solution of  $W_1$ .

The dense representation of  $m(1, T), v_1(1, T), \dots, v_{d+e-1}(1, T)$  can be obtained from that of  $m(Y, T), v_1(Y, T), \dots, v_{d+e-1}(Y, T)$  with  $O((d+e)MN) \subseteq O^\sim(d^{3(d+e-1)})$  operations in  $\mathbb{F}$ . The remaining computations consist of a modular inversion and  $d+e-1$  modular multiplications of univariate polynomials of degree at most  $M$ , which contribute with  $O^\sim(d^{d+e-1})$  additional operations.  $\square$

Step (7) of Algorithm 6.1 has to find a root of  $m_1 \in \mathbb{F}[T]$ , with  $u = \deg m_1 \leq d^{d+e-1}$ . A natural implementation is to find an irreducible factor  $m_2$  of  $m_1$  and use the root  $T \bmod m_2 \in \mathbb{K} = \mathbb{F}[T]/(m_2)$ . While the cost of all other steps can be analyzed for general  $\mathbb{F}$ , this is not clear here. Over a finite field  $\mathbb{F}_q$ , the factorization can be done probabilistically with  $O^\sim(u^{1.5} \log q + u \log^2 q)$  steps (Kedlaya & Umans [24]). For  $\mathbb{F} = \mathbb{Q}$ , efficient factorization algorithms are known, but their cost estimates require a bound on the size of the coefficients of  $m_1$ . Such a bound can be derived along the computations of Algorithm 6.1, but we have not done so. Thus in the following we use some ‘‘cost of root finding at degree  $u$ ’’ for which we have no good estimate in general.

**Lemma 6.10.** *Step (7) of Algorithm 6.1 can be performed with  $O^\sim(d^{2(d+e-1)})$  operations in  $\mathbb{F}$  plus the cost of root finding at degree not more than  $d^{d+e-1}$ .*

*Proof.* For any root  $\eta$  of  $m_1 \in \mathbb{F}[T]$ ,  $\mathbb{K} = \mathbb{F}[\eta]$  has degree at most  $d^{d+e-1}$  over  $\mathbb{F}$ . For any  $j < d+e$ , the same bound holds for the degree of  $w_j \in \mathbb{F}[T]$ , so that all  $w_j(\eta) \in \mathbb{K}$  can be computed with  $O^\sim(d^{d+e-1})$  operations in  $\mathbb{F}$ . Then  $O^\sim(d^{2(d+e-1)})$  operations in  $\mathbb{F}$  are sufficient to test whether  $(g \circ h)(\alpha_j) = \beta_j$  for  $1 \leq j < d+e$ .  $\square$

**Theorem 6.11.** *Let  $d, e \geq 2$ ,  $0 < \epsilon < 1$ ,  $\mathbb{F}$  be a field of characteristic not dividing  $d$  and with at least  $32\epsilon^{-1}(d^2 + de)^2 d^{5(d+e-1)}$  elements,  $\alpha \in \mathbb{F}^{d+e-1}$  generic, and  $\beta \in \mathbb{F}^{d+e-1}$ . Then Algorithm 6.1 returns a solution  $(g, h)$  to Problem 1.1, so*



that  $(g \circ h)(\alpha_j) = \beta_j$  for  $1 \leq j < d + e$ , with probability at least  $1 - \epsilon$ . It uses  $O^\sim(d^{6(d+e-1)})$  operations in  $\mathbb{F}$  plus root finding for degree at most  $d^{d+e-1}$ .

*Proof.* The various estimates in Lemmas 6.3 through 6.10 imply a cost of  $O^\sim(d^{6(d+e-1)})$  operations in  $\mathbb{F}$  plus root finding. For a lucky choice of  $\lambda$ , the output is indeed a solution to Problem 1.1, and the probability for this to happen is at least  $1 - \epsilon$  by Lemma 6.5.  $\square$

## 7. COST OF THE ALGORITHM

We have not tried to optimize the cost of our method. The input to Problem 1.1 consists of  $2(d + e - 1)$  elements of  $\mathbb{F}$ . We do not expect that runtime polynomial in this input size is feasible, since the coefficients of a solution  $(g, h)$  lie in a field of potentially exponential degree.

The more appropriate view seems to consider the geometric variant of Problem 1.1 as stated in Problem 3.3. This involves the incidence variety  $\Gamma_{n,d}$  and its projection  $\pi$  to  $\overline{\mathbb{F}}^{2(d+e-1)}$ . According to (5.2), their geometric degrees are bounded as  $D = \deg \pi \leq \delta = \deg \Gamma_{n,d} \leq d^{d+e-1}$ . In step (2) of Algorithm 6.1, the quantities  $M$  and  $N$  are defined so that  $M > \delta \geq D$  and  $N > 2\delta D$ . These inequalities are all that is being used about  $M$  and  $N$  in the lemmas above, and the algorithm would also work with  $D + 1$  and  $2\delta D + 1$  in their place.

Within the algorithm, we avoid using the values of  $\delta$  and  $D$  because we do not know them. But we could start the algorithm with some small guess at those values and then keep on doubling until we actually find a solution  $(g, h)$ . The running time then is polynomial in  $\delta$ .

More precisely, the runtime of this variant of our algorithm is  $O^\sim((d + e)^\omega D \delta + (D \delta)^\omega + (d + e) D^2 \delta)$  plus root finding for degree at most  $D$ . In particular, Lemma 6.7 can be modified so that for a lucky choice of  $\lambda$ , step (4) can be performed with  $O^\sim((d + e) D^\omega \delta) \subset O^\sim(d^{4(d+e-1)})$  operations in  $\mathbb{F}$ . We briefly discuss below this modification.

The important papers of Lecerf [20] and van der Hoeven & Lecerf [19] present algorithms for solving systems of polynomial equations, with running times  $O^\sim(B^3)$  and  $O^\sim(B^2)$ , respectively, where  $B$  is the system's Bézout number. The values  $D$  and  $\delta$  considered above satisfy  $D \leq \delta \leq B$ . Our approach can approximate  $D$  and  $\delta$  by binary search, as described. In cases where they are considerably smaller than  $B$ , the running time presented here may be smaller than that of the cited works.

The paper of [19] assumes that their elimination encounters only radical ideals and that the polynomial system forms a regular sequence. In our case, the latter requires a saturation with respect to the Jacobian. This has degree up to roughly  $n^{d+e}$ ; see Figure 3.5. If this is added to the equations of Problem 3.3 via the Rabinovich trick, then it will increase the Bézout number by the stated factor. Our approach circumvents this problem by using a starting value for the Newton iteration where the Jacobian does not vanish.

For the characteristic  $p$ , we only demand that it does not divide the degree  $n$  of the target polynomial. [20] is stated only for characteristic zero.

We discuss here [20] and [19], because they yield important progress in general. But it does not seem that they are applicable to our problem.

**7.1. Hermite–Padé approximations to perform step (4) of Algorithm 6.1.**  
In this section we briefly discuss a variant to perform step (4) of Algorithm 6.1.

Recall that, in such a step, for each  $j \in \{1, \dots, d+e-1\}$  we find by binary search the smallest values  $k_j$  of  $\ell$  and, given  $k_j$ , of  $M_j$  for  $M'$  for which

$$(7.1) \quad \sum_{0 \leq i \leq \ell} a_i^{(j)} (L^{(j)})^i \equiv 0 \pmod{(Y^N, \Lambda_j^2)}$$

has a solution  $a_0^{(j)}, \dots, a_\ell^{(j)} \in \mathbb{F}[\Lambda_j][Y]$  with  $\deg_Y a_i^{(j)} \leq \delta$  for  $i < \ell$ ,  $\deg_Y a_\ell^{(j)} < M'$  and  $a_\ell^{(j)}$  monic. For this purpose, we consider (7.1) as a system of linear equations for the coefficients in  $\mathbb{F}[\Lambda_j]/(\Lambda_j^2)$  of the  $a_i^{(j)}$ 's, and compute such a solution  $a_0^{(j)}, \dots, a_{k_j}^{(j)}$  with  $a_{k_j}^{(j)} = 1$  monic of degree  $M'_j$ .

To find such a solution of (7.1), we interpret it as a problem of Hermite-Padé approximation, which can be solved applying an algorithm of Bostan et al. [6], based on fast linear-algebra algorithms for matrices of fixed displacement rank (cf. Bini & Pan [4], Pan [27]). More precisely, for a suitable ordering of the unknowns, the matrix  $B_j$  of system (7.1) is a block-Toeplitz matrix with  $\ell$  blocks (see Cafure et al. [8, Lemma 4.3]; see also [6, Lemma 17]).

From [6, Corollary 1] it follows that, if there exist a nonzero solution of (7.1), then one such solution can be computed with  $O^{\sim}(\ell^{\omega-1}M(\ell M'))$  arithmetic operations in  $\mathbb{F}$  and error probability at most  $1/(2\epsilon)$  for any fixed  $\epsilon > 0$ . The algorithms of [6] work over a field. We apply their results to computations over  $\mathbb{F}((\Lambda_j))$ , but perform calculations in  $\mathbb{F}[\Lambda_j]/(\Lambda_j^2)$ , which is all that we need. This works because our conditions guarantee that all divisions are by series which are invertible in  $\mathbb{F}[\Lambda_j]/(\Lambda_j^2)$ . Thus computing in  $\mathbb{F}[\Lambda_j]/(\Lambda_j^2)$  is sufficient. Taking into account that at most  $2 \log_2(2D)$  values of  $\ell$  and  $2 \log_2(2\delta)$  values of  $M'$  occur for  $1 \leq j \leq d+e-1$ , we have the following result.

**Lemma 7.1.** *For a lucky choice of  $\lambda$ , step (4) of Algorithm 6.1 can be computed with  $O^{\sim}((d+e)D^\omega \delta)$  operations in  $\mathbb{F}$  and error probability at most  $1/(2\epsilon)$  for any fixed  $\epsilon > 0$ .*

Next we analyze the probability of failure of the choice of the vector  $\lambda$ .

**Proposition 7.2.** *Let  $\mathcal{S} \subset \mathbb{F}$  be a finite set with at least  $64\epsilon^{-1}(d+e)D^2(\delta+2)^{\log \delta+3}$  elements. A  $\lambda$  chosen uniformly at random in  $\mathcal{S}^{d+e-1}$  is lucky with probability at least  $1 - 2\epsilon$ .*

*Proof.* Arguing as in Lemma 6.5, we analyze the denominators in  $\mathbb{F}[\Lambda_j]$  which occur when using the algorithm of [6] to solve the Hermite-Padé approximation problem (7.1) for  $1 \leq j \leq d+e-1$ . This algorithm is an adaptation of Kaltofen's *Leading Principal Inverse* algorithm (Kaltofen [22], [23]). The algorithm performs a recursive reduction of the computation of the inverse of a “generic-rank-profile” square input matrix

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}$$

to that of the leading principal submatrix  $A_{1,1}$  and its Schur complement  $\Delta = A_{2,2} - A_{2,1}A_{1,1}^{-1}A_{1,2}$ . The divisions which arise during the execution of this recursive step are related to the computation of  $A_{1,1}^{-1}$  and  $\Delta^{-1}$  and a routine of “compression” (cf. [4, Problem 2.2.11.c]) of the generators of matrices which are obtained as certain products involving  $A_{1,1}^{-1}$ ,  $\Delta^{-1}$ ,  $A_{1,2}$  and  $A_{2,1}$ . The latter in turn requires the computation of the inverses of certain submatrices of the products under consideration.

Each entry of the matrix  $B_j$  of the linear system (7.1) is a coefficient of a power  $(L^{(j)})^i$ , which is therefore a polynomial in  $\Lambda_j$  of degree at most  $i \leq \ell$ . Since the generic-rank-profile matrix  $A$  is obtained by multiplying the matrix  $B_j$  with suitable matrices with entries in  $\mathbb{F}$ , we conclude that the entries of  $A_{1,1}$  are polynomials of  $\mathbb{F}[\Lambda_j]$  of degree at most  $\ell \leq D$ , while the numerators and denominators of the entries of  $\Delta^{-1}$  are polynomials of  $\mathbb{F}[\Lambda_j]$  of degree at most  $D(\delta + 2)$ . Therefore, by a simple recursive argument it can be seen that the numerators and denominators of all leading principal submatrices and Schur complements which are inverted during the algorithm have degrees bounded by  $\mathcal{D} = D(\delta + 2)^{\log \delta}$ . This in turn implies that the denominators arising during the compression routine have degrees bounded by  $3\mathcal{D}\delta$ .

Taking into account that the algorithm of [6] for each value of  $\ell$  and  $M'$  consists of at most  $\log_2 \delta$  recursive steps, and that each recursive step requires the inversion of at most 4 matrices, we conclude that the product of all the denominators arising during the algorithm has degree bounded by  $8\mathcal{D}\delta \log_2 \delta$ . Since at most  $2 \log_2(2D)$  values of  $\ell$  and  $2 \log_2(2\delta)$  values of  $M'$  occur for  $1 \leq j \leq d + e - 1$ , we conclude that the degree of all denominators is bounded from above by  $32(d + e)\mathcal{D}\delta \log_2^3(2\delta)$ .

Finally, according to Lemma 5.5,  $\lambda$  must not annihilate a polynomial  $R^{(4)}$  of degree  $2D^2$ . The statement of the proposition readily follows.  $\square$

We recall from (4.3) and (4.4) that  $D \leq \delta \leq d^{d+e-1}$ . Then from the above estimate, Lemma 7.1 and the error probability  $1/(2\mu)$  of the algorithm underlying Lemma 7.1, we deduce the following result.

**Theorem 7.3.** *Let  $d, e \geq 2$ ,  $0 < \epsilon < 1$ ,  $\mathbb{F}$  be a field of characteristic not dividing  $d$  and with at least  $64\epsilon^{-1}(d + e)D^2(\delta + 2)^{\log \delta + 3}$  elements,  $\alpha \in \mathbb{F}^{d+e-1}$  generic, and  $\beta \in \mathbb{F}^{d+e-1}$ . Then a solution  $(g, h)$  to Problem 1.1 can be computed with probability at least  $1 - \epsilon$ . It uses  $O^\sim((d + e)^\omega D^\omega \delta)$  operations in  $\mathbb{F}$  plus root finding for degree at most  $D$ .*

#### OPEN QUESTION

Our algorithm uses time polynomial in the *geometric input size*, namely the degree of the variety  $\Gamma_{n,d}$  in (3.2) that encodes our problem. The entries of an output  $(g, h)$  lie in an extension field whose degree is bounded by  $D \leq d^{d+e-1}$  and our runtime estimate is polynomial in this bound. It remains open how close this bound is to the actual minimal extension degree required; some experiments might shed light on this, but we have not done this.

One may wonder whether time polynomial in the actual output size of an instance is possible, or whether some particularly succinct representation of the output allows a faster solution. The special cases treated in Section 2 even allow a solution in time polynomial in the size of the input, consisting just of the  $d + e - 1$  field elements that form  $\alpha$  and  $\beta$ . This cannot be expected in general.

Another approach would be to use a numerical instead of our algebraic homotopy. We have not investigated this.

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